Università degli Studi di Catania
Dipartimento di Matematica e Informatica

Tesi di Dottorato in Matematica

A-priori estimates for some classes of elliptic problems

Relatore
Prof. Salvatore A. Marano
Correlatore
Prof. Sunra J.N. Mosconi

Candidata
Greta Marino

XXXI Ciclo

To my Grandma Francesca

"The greatest enemy of knowledge is not ignorance,
it is the illusion of knowledge."
Stephen Hawking

Acknowledgements

I would like to express my sincere gratitude to every people who supported me during the years of my PhD. Above all I am deeply grateful to my family, for their silent and tireless presence, and for sharing with me all the good moments as well as the less good ones.

I also want to thank my supervisor, Prof. S.A. Marano, for all the opportunities he gave me during these years, his encouragements and his confidence in my capacities.

I would like to thank all my friends, both the people that are sharing my way since the years of the school and the people who entered in my life during the course of the time: not in order of importance, my affection is extended to Mimma, Simona, Silvia, Concia, Concy, Andrea, Lorenzo, Nicola, Giorgia, Guido, Patrick.

My last thanks is addressed to Prof. Sunra Mosconi. I am completely grateful to him, because he showed me the way to become a good mathematician and with his example contributed to build the person who I am.

Contents

Introduction

This thesis arises with the purpose of emphasizing some aspects of a powerful tool widely employed in mathematical analysis: the *a-priori estimates*. Indeed, it is well known that a-priori estimates play an important role in the theory of partial differential equations and in the calculus of variations, since they are intimately related with the existence of solutions for a given problem. We will present three of the papers written during this PhD, which are connected by this topic.

Chapter 1 contains some results obtained jointly with Prof. S. Mosconi from the University of Catania and published in the paper [92] on Journal of Differential Equations. The existence of solutions to the equation

$$u'''' + qu'' + F'(u) = 0, \tag{1}$$

where $q \in \mathbb{R}$ and F is a C^2, coercive, and quasi-convex function, that is,

$$F'(t)t \geq 0, \quad \forall\, t \in \mathbb{R}.$$

is investigated. Equation (1) can be classified according to the sign of q. More precisely, we say that (1) is the Extended Fisher-Kolmogorov (EFK) equation if $q \leq 0$, while (1) is the Swift-Hohenberg (S-H) equation if $q > 0$.

Equation (1) has important applications in the real world, since it describes complex patterns in many systems that come from Physics and Mechanics. More precisely, the EFK equation arises as a model equation for certain physical systems that are *bistable* and in the study of phase transitions near singular points (see [41] and [117], respectively). On the other hand, the S-H equation has been widely employed in the description of cellular flows and in the context of lasers (see [130, 30] as well as [81, 133]). But the greatest application of the S-H equation is in the description of the suspension bridges. The idea was proposed by Lazer, McKenna and Walter [77, 95, 96] in order to model a suspension bridge as a vibrating beam supported by cables, with a nonlinear response to loading and a constant weight per unit length due to gravity. All these topics represent the first part of Chapter 1.

The second part focuses instead on our results [92], which provide an answer to some questions posed in [79]. We obtained some nonexistence results for the EFK equation, see Theorems 1.6.1 and 1.6.3, and some existence results for the S-H equation, which also give the exact parameter range for which the equation has a nontrivial bounded solution, see Theorems 1.6.4 and 1.6.5.

The last part of Chapter 1 deals with the asymptotic behavior of the periodic solutions obtained in the aforementioned results. More precisely, we show that, under suitable conditions on the function F, every solution of equation (1) is such that $\|u\|_\infty \to 0$, as $q \to 0$; see Corollary 1.6.1 and Theorem 1.6.7, respectively.

Here the importance of the a-priori estimates lies in the fact that they allow us to obtain some qualitative and global properties of solutions.

Chapter 2 is based on the results obtained in [93] written in collaboration with Prof. P. Winkert from the University of Technology Berlin, Germany, and already published on Nonlinear Analysis. In this work, we studied the global boundedness of solutions to the following boundary value problem

$$\begin{aligned} -\operatorname{div}\mathcal{A}(x,u,\nabla u) &= \mathcal{B}(x,u,\nabla u) \qquad && \text{in } \Omega, \\ \mathcal{A}(x,u,\nabla u)\cdot\nu &= \mathcal{C}(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where $\nu(x)$ denotes the outer unit normal of Ω at $x \in \partial\Omega$, and $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ satisfy suitable p-structure conditions which can allow critical growth, in Ω as well as on $\partial\Omega$.
Boundary value problems with critical exponent have always represented an important task to overcome. Indeed, since the embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is not compact, this in turn implies that the functional associated to a prescribed problem does not satisfy the Palais-Smale condition. Hence, there are serious difficulties when trying to find its critical points by standard variational methods. Several authors have developed different strategies in order to overcome these difficulties, and a detailed description of them can be found in Section 2.2.
The main motivation for studying critical problems like (2) stems from the fact that they arise from some variational problems in Geometry and Physics where lack of compactness also occurs. In particular, (2) can be seen as a generalization of the classical Yamabe problem

$$-\Delta u = f(x)u + h(x)u^{\frac{N+2}{N-2}}, \tag{3}$$

where f and h are smooth functions. It is well known that there is no stable regularity theory for solutions of (3), which reflects the difficulty of the Yamabe problem. Nevertheless, it was proven by Trudinger [137] that any $W^{1,2}(\Omega)$ solution of (3) is in fact smooth, but its regularity estimate depends on the solution itself. In this spirit, the main result of Chapter 2, Theorem 2.6.1, can be seen as a generalization of Trudinger's work. Theorem 2.6.1, whose proof is performed in Subsection 2.6.2, contains two important assertions. First of all, it states that every solution $u \in W^{1,p}(\Omega)$ of problem (2) is in $L^r(\overline{\Omega})$ for every $r < +\infty$, and then that u is actually in $L^\infty(\overline{\Omega})$, with a bound which depends on the given data and on the solution itself. The main tool which allowed us to obtain these results is a modified version of Moser's iteration technique, which in turn is based on the books [44, 129], and which is also briefly presented in Section 2.5.
Chapter 2 ends with some regularity results. We proved that, with some additional assumptions on the functions $\mathcal{A}$ and $\mathcal{C}$, every weak solution of (2) is actually in $C^{1,\beta}(\overline{\Omega})$, for some $\beta \in (0,1)$, see Theorem 2.6.2.
The importance of the a-priori estimates is emphasized not only in Theorem 2.6.1, but rather in the fact that the a-priori bound of a given solution to problem (2) directly entails regularity considerations ensuring that $u \in C^{1,\beta}(\overline{\Omega})$.

Chapter 3 is based on the results obtained in [91], written in collaboration with Prof. S.A. Marano and Prof. A. Moussaoui from the University of Mira Bejaia, Algeria. It focuses on an existence result for the following singular system

$$\begin{aligned} &-\Delta_{p_1} u = a_1(x) f(u,v) \quad \text{in } \mathbb{R}^N, \\ &-\Delta_{p_2} v = a_2(x) g(u,v) \quad \text{in } \mathbb{R}^N, \\ &\ u, v > 0, \quad u, v \to 0 \text{ as } |x| \to +\infty. \end{aligned} \tag{4}$$

Here with the word singular we mean an equation which contains terms that could go to $+\infty$ as variables approach zero. And system (4) is effectively singular since, according to our hypotheses, the functions f and g have a growth that involves also negative exponents.
Singular semilinear systems in a bounded domain were introduced by Gierer and Meinhardt [57] as a mathematical model in biochemical processes. The non singular version has instead been widely employed in the study of interacting populations. These interactions, whose description is contained in the first part of the chapter, are of three types: the predator-prey model, which occurs if the growth rate of one population is decreasing while the other is increasing with respect to the first; the competition model, which occurs if the growth rate of each population is decreasing with respect to the other; finally, the mutualism, which happens when each population's growth rate is enhancing. All these situations are described by systems of ordinary differential equations or partial differential equations that also take into account the tendency of each population to spatially diffuse.
The second part of the chapter treats singular systems, starting from the semilinear case in a bounded domain, see Subsection 3.4.1, and then concentrating on the quasilinear case, see Subsection 3.4.2. Since variational methods do not work, different other techniques, mainly based on fixed point arguments in a sub-supersolution setting, were developed. Of course these problems can be generalized considering the case when $\Omega = \mathbb{R}^N$: the semilinear case was treated for example in [101], while the quasilinear case, to the best of our knowledge, has never been studied in the literature until our work [91], which is contained in Section 3.5.
Here we assume some structure conditions on the functions f and g which do not allow to reduce system (4) to the Gerier-Meinhardt's type. The main idea for solving (4) is to perturb the system with a parameter $\varepsilon > 0$, and then to apply Schauder's fixed point theorem in order to get a solution $(u_\varepsilon, v_\varepsilon)$, for every $\varepsilon > 0$. Finally, letting $\varepsilon \to 0^+$ yields a weak solution (u, v) of (4).
Here, the a-priori estimates play a crucial role, because it is only once we know that a solution (u, v), as well as $(u_\varepsilon, v_\varepsilon)$, is uniformly bounded that we can apply some comparisons which directly lead to the existence results.

All the chapters are structured in a similar way. We first introduce the concrete problems which inspired our works. Then, after a description of some known results from the literature, we present the results obtained in our papers. Finally, some open problems and further possible developments are listed.

Chapter 1

Model equations in physical pattern formation

1.1 Introduction

One of the most interesting aspects of the complex dynamics that governs natural phenomena is the occurrence of instabilities and symmetry breakings that lead to the formation of coherent spatio-temporal structures on macroscopic scales. When a physico-chemical system is maintained far from its thermal equilibrium by the application of external constraints, it may go through a succession of instabilities that induce various types of spatio-temporal pattern formation. These phenomena are studied using the methods of nonlinear dynamics and instability theory, because such a organization seems to be related to technological problems that arise, among others, in physics, chemistry and nonlinear optics. Since these systems are described by nonlinear partial differential equations, it is very difficult to solve these problems because of the impossibility to obtain analytical solutions. However, in the 1970s general techniques, based on the analogies between phase transitions and critical phenomena, were developed. In particular, the fact that similar phenomena appear in very different systems (such as spiral waves in chemical systems, cardiac activity, hydrodynamical instabilities in liquid crystals) shows that they are not induced by the microscopic properties of the system but they are triggered by collective effects including a large number of individuals (atoms, molecules, cells).

In order to achieve a better understanding of the dynamical behavior of systems far from equilibrium, well-chosen model equations have been proposed. Even if they are often simpler than the full equations describing those systems, they allow us to underline the mechanisms that are responsible for the formation and evolution of complex patterns. Classical model equations are typically second-order PDEs. One of the most important is the widely studied Fisher-Kolmogorov equation, a nonlinear second-order diffusion equation proposed in 1937 as a model for the interaction of dispersal and fitness in biological populations.

Our interest is instead concentrated in a series of higher-order partial differential equations that have been taken as model in the study of pattern formation in systems from physics and mechanics. For special classes of solutions such as stationary solutions or traveling waves, many of these equations reduce to a simpler one, of the form

$$\frac{d^4u}{dx^4} + q\frac{d^2u}{dx^2} + f(u) = 0. \tag{1.1.1}$$

Here q is a real-valued eigenvalue parameter which measures the tendency of the equation to form complex patterns, and f is a given function. Equation (1.1.1), and the properties of its bounded solutions for different values of q and functions f, are the core of this chapter.

1.2 Model equations

Well-chosen model equations have always played an important role in applied mathematics, because of their presence in a great variety of physical, chemical and biological systems. Classical examples are the heat equation, the wave equation and the Laplace equation, which describe processes like diffusion, dispersion and wave propagation, and also give their mutual interactions and their quantitative description. They are typically linear second-order partial differential equations. However, since many problems in the sciences and in engineering are intrinsecally nonlinear, it became necessary to introduce nonlinear generalizations.

1.2.1 Second-order model equations

A very well-known example from the literature is the *Fisher-Kolmogorov (FK) equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^3. \tag{1.2.1}$$

It was originally proposed to study the fronts that arise in population dynamics. A front is a propagating interface between two different steady states and can be viewed as a balance between diffusive forces coupling different points in the field, and reactive forces which move the system from unstable to stable states. At first sight, front propagation through unstable states might seem to be an esoteric subject. In reality, however, there are many examples where this phenomenon is an essential element of the dynamics. For example, fronts naturally arise in convectively unstable systems, in which a state is unstable, but perturbations are convected away faster than they grow out.

From a qualitative point of view, equation (1.2.1) exhibits an unstable uniform state, $u = 0$, and two stable uniform states, $u = \pm 1$. Fronts that oscillate around zero exist, but they are considered unphysical, since they represent negative population densities. Consequently, interest is focused on strictly non-negative solutions. Fronts in the FK equation connect the unstable to the stable state, and move in such a way as to destabilize the unstable state.

The FK equation is often called the *real Ginzburg-Landau (GL) equation*, since it is the real version of the complex GL equation, an envelope equation that describes the dynamics of wave envelopes near transition in hydrodynamic systems.

Another second-order model equation of interest is the following, known as the *sine-Gordon (sG) equation*

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \sin u.$$

It is a nonlinear hyperbolic partial differential equation in 2 dimensions involving the d'Alembert operator and the sine of the unknown function. It was originally introduced by E. Bour [21] in the study of surfaces of constant negative curvature and rediscovered by Frenkel and Kontorova [50] in their study of crystal dislocations. It attracted a lot of attention in the 1970s because of the presence of soliton solutions, and was also widely used in studies of nonlinear wave propagation, see [45, 119].

1.2.2 Higher-order model equations

The equations introduced above are not able to describe complex patterns for which equation (1.1.1) plays an important role. A typical example of such a pattern is the phenomenon of *localized buckling* in mechanics, which happens when the deflections are confined to a small portion of the otherwise unperturbed material. To understand the formation of complex spatial and temporal patterns, higher-order scalar model equations and systems of equations have been proposed. Below we list some of these equations that are of interest for us, beginning with the one that can be considered their prototype.

Elastic beam

The mechanics of solid bodies, regarded as continuous media, forms the content of the *theory of elasticity*, whose basic equations were established in the 1820s by Cauchy and Poisson. Under the action of applied forces, solid bodies exhibit deformation to some extent, and so change in shape and volume. The three-dimensional equations of a continuous solid elastic medium vibrations are of a great complexity and in general cannot be solved analytically. However, elastic solids present geometrical characteristics which simplify the mathematical analysis of their vibrations. These simplifications have led to the theories of beams, plates and shells. In particular, the theory of beams consists of constructing one-dimensional models and in this sense represents the simplest continuous media. This simplicity is extremely useful since it leads to obtain analytical solutions of the problem equations and, consequently, to study the vibratory phenomena in a comprehensive fashion. Research of the basic vibratory phenomena results in the identification of three elementary movements: longitudinal vibrations, vibrations of torsion and bending vibrations. The study of coupled longitudinal movements, torsion and bending is possible, but with an increased difficulty of resolution. Probably one of the simplest equation for an elastic beam is the following

$$\frac{\partial^2 u}{\partial t^2} + \kappa^2 \frac{\partial^4 u}{\partial x^4} = 0,$$

where κ is characteristic of the given bar.

The Extended Fisher-Kolmogorov (EFK) equation

This equation, which can be regarded as a natural extension of the FK equation (1.2.1), was proposed in 1988 by Dee and van Saarloos [41] during their studies on wave propagation, and models physical systems that are *bistable*

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3, \qquad \gamma > 0. \tag{1.2.2}$$

Indeed, as in the FK equation, the EFK equation has two uniform states $u(x) = \pm 1$ which are stable, separated by a third uniform state $u(x) = 0$ which is unstable. The choice $\gamma > 0$ is dictated by the physical requirement that the model is stable at short wavelengths, but otherwise the fourth-order spatial derivative does not dramatically alter the qualitative features of the homogeneous states. Indeed, the $u = 0$ state remains unstable to long-wavelength perturbations, while the other states remain absolutely stable.

The EFK equation also arises in the study of phase transitions near singular points, the so-called Lifshitz points (LP for short). It can occur in a variety of different systems, including magnetic compounds and alloys, liquid crystals and charge-transfer salts. Finally, equation (1.2.2) arises as amplitude equation at the onset of instability near certain degenerate states. In [117] it was shown, through numerical simulations of the full reaction-diffusion system, that the behavior of small perturbations near the degeneration is best described by the EFK equation, as opposed to the classical Ginzburg-Landau model.

The Swift-Hohenberg (S-H) equation

It was first proposed by Swift and Hohenberg in order to study the effects of thermal fluctuations on a fluid near the Rayleigh-Bénard instability [130]

$$\frac{\partial u}{\partial t} = -(1 + \frac{\partial^2}{\partial x^2})^2 u + \kappa u - u^3, \qquad \kappa \in \mathbb{R}. \tag{1.2.3}$$

Consider a horizontal layer of fluid in which an adverse temperature gradient is maintained by heating the underside. We say that the temperature gradient is adverse because, on account of thermal expansion, the fluid at the bottom will be lighter than the fluid at the top; and this top-heavy arrangement is potentially unstable. Because of this instability there will be a tendency on part of the fluid to redistribute itself and remedy the weakness in its arrangement. However, this natural tendency will be inhibited by its own viscosity. In other words, we expect that the adverse temperature gradient must exceed a certain value before the instability can manifest itself. The earliest experiments to demonstrate in a definitive manner the onset of thermal instability in fluids are those of Bénard in 1900, though the phenomenon of thermal convection itself had been recognized earlier by Count Rumford (1797) and James Thomson (1882). The principal facts they established are the following; first, a certain critical adverse temperature gradient must be exceed before instability can set in; and second, the motions which ensue on surpassing the critical temperature gradient have a stationary cellular character. In a fundamental paper [116], Lord Rayleigh showed that what decides the stability or otherwise of a layer of fluid heated from below is the numerical value of the non-dimensional parameter

$$R = \frac{g\alpha\beta}{\kappa\nu} d^4,$$

often called the *Rayleigh number*. Here g denotes the acceleration due to gravity, d the depth of the layer, β the uniform adverse temperature gradient, and α, κ and ν are the coefficients of volume expansion, thermometric conductivity and kinematic viscosity, respectively. Rayleigh further showed that instability must set in when R exceeds a certain critical value R_c; and, when it happens, a stationary pattern of motions must come to prevail.

Equation (1.2.3) has also been used by Pomeau and Manneville [111] to study the phenomenon of wavenumber selection in cellular flows, which follows from the breaking of translational invariance in large but finite structures. Finally, several authors adapted this equation in the context of lasers, see [81, 133].

The Suspension Bridge equation

This equation was proposed by Lazer, McKenna and Walter [77, 95, 96] and models a suspension bridge as a vibrating beam supported by cables, which has a nonlinear response to loading and a constant weight per unit length due to gravity. The unknown $u(x,t)$ measures deflection from the unloaded state and is therefore applicable to vertical oscillations

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} + (u+1)_+ - 1 = 0. \tag{1.2.4}$$

As written in [77], much of interest in this field started after the breakdown of the Tacoma Narrows suspension bridge [5], which was first subject to large-scale oscillations, followed by the collapse of the structure.[1] The standard explanation of this phenomenon was that the bridge behaves like a particle of mass one at the end of the spring, with spring constant k, which is subject to a forcing term of frequency $\mu/2\pi$. If μ is very close to $\sqrt{k}$, then large oscillations result. If μ is not, then it does not. Accordingly, the forcing term came from a train of alternating vortices being shed by the bridge as the wind blew past it. The frequency just happened to be at a value very close to the resonant frequency of the bridge. Thus, even though the magnitude of forcing term was small, the phenomenon of linear resonance was enough to explain the large oscillation and the eventual collapse of the bridge. Anyway, this explanation was not persuasive.

As made clear in [5, 18], suspension bridges have a history of large-scale oscillation and catastrophic failure under high and even moderate winds. Earlier bridges, such as the Bronx-Whitestone Bridge or the Golden Gate Bridge, had shown oscillatory behavior due to the action of wind. What distinguished the Tacoma Narrows was the extreme flexibility of its roadbed. This resulted a pronounced tendency to oscillate vertically, under widely differing wind conditions. The bridge was also affected by another type of oscillation just prior to the collapse: a pronounced torsional mode observed after the bridge went into large vertical motion. Furthermore, a wind-tunnel study of a scale model of the Tacoma Narrows Bridge in a variety of wind conditions showed that, when attempting to model large amplitude oscillation, the behavior is almost perfectly linear, see [5, Appendix VIII].

Thus, several interesting questions from the mathematical point of view arise. For example, to understand what in the nature of suspension bridges makes them so prone to large-scale oscillation; to find an explanation of the fact that the bridge would go into large oscillation under the impulse of a single gust, or would remain motionless in strong winds; to explain how the large vertical oscillations could rapidly change to torsional; to study the existence of the traveling waves; to get a formal description of why the motion is linear over small to medium range oscillation. It is important to observe that the current explanation of these phenomena was highly incorrect until the work of Lazer and McKenna [77], who constructed the right mathematical model, proving that what distinguishes suspension bridges is their fundamental nonlinearity. The restoring force due to a cable is such that it strongly resists expansion, but does not resist compression. Thus, the simplest function to model this force would be a constant times u, the expansion, if u is positive, but zero if u is negative, corresponding to compression, see equation (1.2.3).

[1] Tacoma Narrows Bridge collapse, https://www.youtube.com/watch?v=nFzu6CNtqec.

Water waves

The behavior of steady nonlinear water waves on the surface of an inviscid heavy fluid layer has received much attention during the past century, both from the mathematical and from the physical side. Some interesting problems are, for example, the existence of solitary and cnoidal (periodic) waves in the presence of surface tension or the description of the reaction of a fluid to a localized pressure distribution moving over its surface with constant speed.

In its long history, the analysis of nonlinear surface waves has been promoted by scientists of various backgrounds, and a vast literature is available for the unforced case, which happens when the pressure at the surface is constant, see [126, 148, 152, 153]. On the other hand, the resonant case, which occurs when the pressure speed coincides with the critical wave speed, became of particular importance and difficulty.

In [73], nonlinearly resonant water waves are analyzed with the only assumption of moderate wave amplitudes. They reduced to waves in two dimensions which, after a suitable rescaling, lead to the following single fourth-order ordinary differential equation

$$u'''' + Pu'' + u - u^2 = 0. \tag{1.2.5}$$

Here the function $u(x)$ is related to $\eta(x) - 1$, where η is the dimensionless depth of the water, while the coefficient P is a negative constant.

Equation (1.2.5) has been the object of much recent study (see [6, 26, 66, 67] and the references therein). In particular, in [6] the existence of homoclinic orbits connecting the zero equilibrium of (1.2.5) is studied, and they can be interpreted as indicating the presence of spatially localized buckling.

The nonlinear Schrödinger equation

It is well-known that the dynamics of optical fibers is governed by the nonlinear Schrödinger equation (NLSE). By contrast to the usual theory of evolutionary PDEs, in the NLSE the evolution variable is the 'space' variable, namely the longitudinal coordinate of the fiber. In [2] a generalized NLSE with a negative fourth-order dispersion term is investigated. It can be obtained under the usual assumptions of the absence of derivative nonlinearities which, after a suitable rescaling, lead to the following equation

$$i\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial t^2} - \frac{\partial^4 v}{\partial t^4} + |v|^2 v = 0. \tag{1.2.6}$$

Stationary pulse-like solutions of equation (1.2.6) have the form

$$v(x,t) = u(k,t)e^{ikx}, \tag{1.2.7}$$

where k is the soliton propagation constant and $u(s,t)$ is a real function of its parameters. Substitution of the ansatz (1.2.7) into equation (1.2.6) gives

$$u'''' - u'' + ku - u^3 = 0, \tag{1.2.8}$$

where k is now the only parameter of the problem. Equation (1.2.8) is a nonlinear ordinary differential equation of fourth order, and its localized solutions give the stationary soliton-like solutions of equation (1.2.6). From the mathematical point of view, the treatment of the Schrödinger equation (even the linear one) may be delicate since such equation possesses a mixture of the properties of parabolic and hyperbolic equation. In the NLSE for optical fibers, because of attenuation, there is not conservation of energy.

1.3 The reduction to an ODE

When we look for special classes of solutions, such as stationary solutions or traveling waves, all the aforementioned equations reduce to an ordinary differential equation of the type (1.1.1), which can be written in the equivalent form

$$u'''' + qu'' + F'(u) = 0. \tag{1.3.1}$$

Here with primes we mean the differentiation with respect to the variable x. The function F is often called the *potential* although, according to the usual terminology in classical mechanics, its opposite $-F$ should be called the potential.

After a suitable rescaling, we can write the stationary version of all the equations (1.2.2)-(1.2.5), (1.2.8) in the form (1.3.1). More precisely, for $q = -1/\sqrt{\gamma}$ and $F(u) = \frac{1}{4}(1-u^2)^2$, we obtain the Extended Fisher-Kolmogorov equation

$$u'''' + qu'' + u^3 - u = 0. \tag{1.3.2}$$

When $k > 1$, for $q = 2/\sqrt{k-1}$ and $F(u) = \frac{1}{4}(1-u^2)^2$, we have the Swift-Hohenberg equation. Traveling wave solutions $u(x,t) = w(x-ct)$ of equation (1.2.4) lead to equation (1.3.1) for $q = c^2$. When $q = P$ and $F(u) = \frac{u^2}{2} - \frac{u^3}{3}$, we have the water waves equation (1.2.5), while for $q = -1/\sqrt{k}$ and $F(u) = -(\frac{u^2}{2} - k^2)$ we have the nonlinear Schrödinger equation (1.2.8).

Equation (1.3.1) can be classified according to the sign of the parameter q. When $q \leq 0$ we say that it is of the Extended Fisher-Kolmogorov-type, while it is of the Swift-Hohenberg-type if $q > 0$. We can further classify equation (1.3.1) according to the choice of the potential F. There are several possibilities.

1. F is a double-well coercive potential. In this case F exhibits two zeros, or equilibria, $u(x) = x_{1,2}$ at the same energy level. Potentials of this type appear, for example, in the EFK equation or in the S-H equation when $k > 1$. When F is a multi-well potential, equation (1.3.1) was proposed in [59] as a model for ternary mixtures in order to overcome the defects of the classical Ginzburg-Landau models, which rules out observed transitions between non consecutive equilibria.

2. F is a single-well coercive potential. It has only one zero and goes to $+\infty$ both at $+\infty$ and $-\infty$. It appears in equation (1.2.4) or in (1.2.3) when $k < 1$ (see [106]), and in general in model equations for suspension bridges (see [31]).

3. F is an anticoercive potential. It is such that its opposite $-F$ is coercive. We can observe it, for example, in the nonlinear Schrödinger equation (1.2.6).

4. F is one-sided coercive potential. It is such that $F(t) \to +\infty$ for $t \to +\infty$ and $F(t) \to -\infty$ for $t \to -\infty$, or vice-versa. Such a potential is typical of the water waves equation (1.2.5).

Both of these classifications can be mixed, and thus one can obtain an EFK equation with a single-well coercive potential, and so on.

There are two important functionals associated to (1.3.1). First, when we multiply the equation by u' and integrate, we obtain the *energy* or *Hamiltonian*

$$\mathcal{E}(u) := u'u''' - \frac{1}{2}u''^2 + \frac{q}{2}u'^2 + F(u). \tag{1.3.3}$$

The energy is a conserved quantity along orbits of (1.3.1). This means that, if u is a solution of (1.3.1), then

$$u'u''' - \frac{1}{2}u''^2 + \frac{q}{2}u'^2 + F(u) := \text{ constant} := E. \tag{1.3.4}$$

Second, the *Lagrangian action* associated with this Hamiltonian

$$\mathcal{J}(u) := \int \left(\frac{1}{2}|u''(x)|^2 - \frac{q}{2}|u'(x)|^2 + F(u(x)) \right) dx.$$

The solutions of (1.3.1) correspond to critical points of the action $\mathcal{J}(u)$ and vice-versa. The domain of integration depends on the type of solution under investigation. We will go into more details of this *variational* structure in Section 1.4.3.

In many physical problems one is primarily interested in the large-time behavior of solutions of the evolution equation (1.3.1). Here the *attractor* of the dynamical system defined by the equation plays an important role. In bounded domains the attractor often consists of stationary solutions. In unbounded domains, if the equation is invariant with respect to spatial translations, the attractor may also contain traveling wave solutions.

The class of bounded solutions of (1.3.1) is particularly relevant, since most of patterns defined on infinitely extended domains correspond to its uniformly bounded solutions. We will denote this class by $\mathcal{B}$. Since the mid-1990s, it has become evident that the structure of $\mathcal{B}$ can be very rich indeed, and includes a wealth of solutions of different nature, depending both on the function F and on the value of q. For example, consider the simplest linear equation

$$u'' + \lambda u = 0, \qquad \lambda \in \mathbb{R}.$$

In this case, the set $\mathcal{B}$ is very restricted: if $\lambda < 0$, it consists of the trivial solution only; if $\lambda = 0$, it consists of the constants; finally, if $\lambda > 0$, it consists of the linear combination of $\sin(x\sqrt{\lambda})$ and $\cos(x\sqrt{\lambda})$.

When the nonlinearity is introduced, the set $\mathcal{B}$ becomes richer. Below we list some of the most important type of solutions contained in $\mathcal{B}$ that are object of our interest. For the remaining part of this section, we suppose that the nonlinearity F has two distinct equilibria $u(x) = x_{1,2}$, at the same energy level, as in the case of equation (1.3.2).

1. Heteroclinic and homoclinic solutions. From a qualitative point of view, it is interesting to study the connections between the equilibria by trajectories of solutions of the equation. These are called *homoclinic* or *heteroclinic* solutions, sometimes *pulses* or *kinks*, according to whether they describe a loop based at one single equilibrium or they "start" and "end" at two distinct equilibria. More specifically, we say that u is a *heteroclinic solution* of (1.3.1) connecting x_1 to x_2 if

$$\lim_{x\to-\infty}(u, u', u'', u''')(x) = (x_1, 0, 0, 0) \quad \text{and} \quad \lim_{x\to+\infty}(u, u', u'', u''')(x) = (x_2, 0, 0, 0). \tag{1.3.5}$$

Of course, u is a heteroclinic solution connecting x_2 to x_1 if the previous relations hold with the limes to $+\infty$ and $-\infty$ inverted. We say that u is a *homoclinic solution* of (1.3.1) if

$$\lim_{x\to\pm\infty}(u, u', u'', u''')(x) = (x_1, 0, 0, 0) \qquad (\text{or } (x_2, 0, 0, 0)).$$

2. Periodic solutions. The existence of this type of solution corresponds to the formation of spatially periodic patterns in systems described by (1.3.1). Periodic solutions can be classified according to the *energy* (1.3.4) or according to their period, $2T$.

3. Chaotic solutions. They present an infinite number of jumps between the constant solutions and possesses between successive jumps a prescribed number of small oscillations around x_1 and x_2. Most of them are *multibump* solutions, whose graphs have more than one critical point in a half-period if they are periodic and between tails if they converge to one or both the equilibria.

1.3.1 Linearization

The set $\mathcal{B}$ has been the object of much research in the last years. The reason is that it is strongly affected by q, since the nature of the equilibria of F changes at the critical values of q. Therefore, this parameter has a key role in the analysis of the behavior of solutions of equation (1.3.1) (see [103, 104, 105, 106]). Suppose that $F'(0) = 0$ and $F''(0) = k > 0$. Linearizing near zero we obtain

$$u'''' + qu'' + ku = 0, \tag{1.3.6}$$

and so the associated characteristic equation $\lambda^4 + q\lambda^2 + k = 0$, with eigenvalues

$$\lambda = \pm\sqrt{\frac{-q \pm \sqrt{q^2 - k}}{2}}.$$

When $q \leq -\sqrt{k}$, the four eigenvalues are real and so $u = 0$ is a *saddle-node*. For $q \in (-\sqrt{k}, \sqrt{k})$, they are all complex with non-vanishing real parts. In this case the equilibrium is called *saddle-focus*. When $q = \sqrt{k}$ there is a reversible Hopf bifurcation, and all four eigenvalues become purely imaginary. They remain imaginary for all $q > \sqrt{k}$. In this case, $u = 0$ is a *center*.

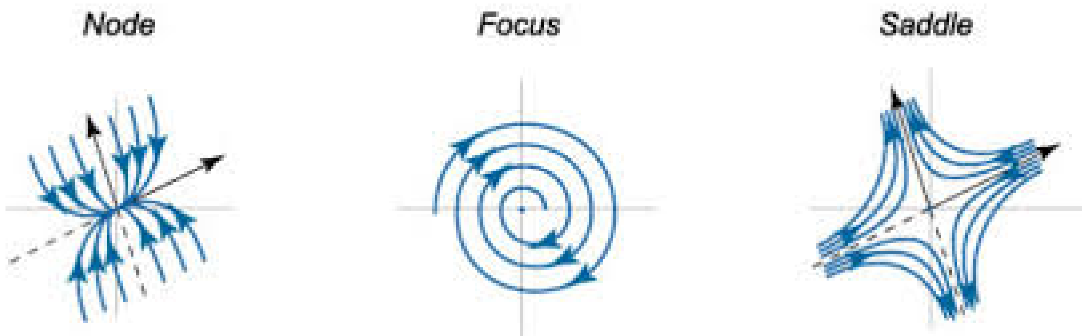

Figure 1.1: `Equilibria`

The behavior of the solutions of (1.3.6) provides important informations concerning the solutions of the nonlinear equation. For example, in the saddle-focus case, it is well known that, under suitable smoothness assumptions, the nonlinear flow and the flows defined by the linearization are conjugate in a neighborhood of the equilibrium (see [63]). Consequently, in this case, as well as the saddle-node one, the solutions of (1.3.1) inherit some properties of the solutions of (1.3.6) when they are close to $u = 0$. For example, we easily obtain a qualitative description of any heteroclinic at $\pm\infty$. Indeed, when 0 is a saddle-node, the solutions of (1.3.6) that vanish at $+\infty$ or $-\infty$ are monotone, while in the saddle-focus case, they do oscillate around zero.

1.4 Methods

Different methods have been developed to study equation (1.3.1). Since it contains only even order derivatives and is autonomous, this system is both reversible and Hamiltonian. This perspective then allows one to apply general results about dynamical systems. Thus we can analyze in detail the bifurcation and structure of different periodic solutions and homoclinic orbits near critical points (see [8, 139]). Moreover, results of Devaney as well as Vanderbouwhede and Fiedler [43, 140] can be used to find families of periodic solutions on the basis of the existence of homoclinic orbits. An important restriction of these methods is that they are in some sense local, valid either near an equilibrium point or near a homoclinic or a heteroclinic orbit.

In order to derive *global* results, a variety of alternative approaches have been developed. Below we briefly describe some of them.

1.4.1 Topological shooting

What has come to be called the shooting method has its origin in a more sophisticated technique, due mainly to Ważewski [145]. It makes use of a topological lemma that is related, in $\mathbb{R}^N$, to Brouwer's fixed point theorem. Shooting may be thought of as including Ważewski's method, but also simpler topological arguments involving only connectedness (see [38, Section 2.5]). It is possible to use it more broadly, for each argument in which a boundary value problem is shown to have a solution by considering the topology of the space of initial condition. In other words, we solve a boundary value problem by reducing it to an initial value problem. The main idea of a classical shooting method is to look at the way solutions change with respect to initial conditions (taken as parameters) at some fixed initial point. Roughly speaking, we 'shoot' out trajectories in different directions until we find a trajectory that has the desired boundary value. Consider for example the following *boundary value problem* (see [106] for further details)

$$\begin{aligned} &u'' - u = 0 && \text{on } (0,1), \\ &u(0) = 0 \quad \text{and} && u(1) = 3, \end{aligned} \tag{1.4.1}$$

and suppose that we wish to prove that there exists a solution of it. There are many ways to do this. In the method based on topological shooting, one replaces all the conditions at one of the boundary points by additional conditions at the other boundary point, so that near this point the equation has a unique solution which satisfies the combined old and new boundary conditions. For instance, here we can replace the condition at $x = 1$ and impose a slope u' at $x = 0$. We are then left with the *initial value problem*

$$\begin{aligned} &u'' - u = 0 && \text{on } (0,1), \\ &u(0) = 0 \quad \text{and} && u'(0) = \alpha, \end{aligned} \tag{1.4.2}$$

where $\alpha \in \mathbb{R}$ is a parameter which we are free to choose. By standard ODE theory [36], problem (1.4.2) has a unique local solution $u(x, \alpha)$, for every $\alpha \in \mathbb{R}$. The question is now whether there exists a value α^* such that $u(\,\cdot\,,\alpha^*)$ exists on $[0,1]$ and $u(1,\alpha^*) = 3$.

Since the differential equation is linear, for each $\alpha \in \mathbb{R}$, the solution $u(x,\alpha)$ can be continued all the way to $x = 1$. Thus, $u(1,\alpha)$ is well defined on $\mathbb{R}$, and we can introduce the sets

$$\mathcal{S}^+ = \{\alpha \in \mathbb{R} : u(1,\alpha) > 3\} \quad \text{and} \quad \mathcal{S}^- = \{\alpha \in \mathbb{R} : u(1,\alpha) < 3\}.$$

Evidently, if $\alpha \in \mathcal{S}^+$, the solution hits the line $\{x = 1\}$ in the (x,u)-plane too high, while if $\alpha \in \mathcal{S}^-$, then the solution hits the line $\{x = 1\}$ too low.

Suppose now that one has shown that $\alpha_- \in \mathcal{S}^-$, $\alpha_+ \in \mathcal{S}^+$ and that the function $\Phi(\alpha) := u(1,\alpha)$ is continuous on the interval $[\alpha_-,\alpha_+]$. Then plainly the sets $\mathcal{S}^+$ and $\mathcal{S}^-$, restricted to $[\alpha_-,\alpha_+]$, are open and the existence of an $\alpha^* \in [\alpha_-,\alpha_+]$ where $\Phi(\alpha^*) = 3$ follows. Of course, the continuity of Φ immediately implies the existence of α^*. Notice that the previous argument says nothing about the *number* of solutions of problem (1.4.1).

In [103, 104, 106], Peletier and Troy developed a topological shooting method especially adapted to track monotone heteroclinics for the EFK equation (1.3.2). It turns out that their approach works well for $q \le -\sqrt{8}$, the saddle-nodes case.

1.4.2 Hamiltonian method

The evolution of many conservative systems can be described by Hamilton's equations:

$$\begin{aligned} \dot{r}_i &= \frac{\partial H}{\partial p_i}(r,p), && 1 \le i \le N, \\ \dot{p}_i &= -\frac{\partial H}{\partial r_i}(r,p), && 1 \le i \le N. \end{aligned} \tag{1.4.3}$$

Here, (r,p) belongs to $\mathbb{R}^N \times \mathbb{R}^N$, the so-called phase space, and N is the number of degrees of freedom. The first N components $r = (r_1,\dots,r_N)$ represent position variables, and the last N ones $p = (p_1,\dots,p_N)$ momentum variables. The function $H\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, the Hamiltonian, represents the energy of the system. It is an immediate consequence of equation (1.4.3) that H is an integral of motion, i.e., $H(r(t),p(t))$ is constant along any solution of (1.4.3). In other words, it holds

$$\begin{aligned} \frac{d}{dt}[H(r(t),p(t))] &= \sum_{i=1}^N \frac{\partial H}{\partial r_i}\dot{r}_i + \sum_{i=1}^N \frac{\partial H}{\partial p_i}\dot{p}_i \\ &= \sum_{i=1}^N \frac{\partial H}{\partial r_i}\frac{\partial H}{\partial p_i} + \sum_{i=1}^N \frac{\partial H}{\partial p_i}\Big(-\frac{\partial H}{\partial r_i}\Big) = 0, \end{aligned}$$

for every solution of (1.4.3), and so the energy is a conserved quantity.

In many problems that arise from nonlinear mechanics, as the modeling of nonlinear water waves, the Hamiltonian system has an energy functional which is given by

$$H(r,p) = \frac{1}{2}(Sp,p) + V(r), \qquad (r,p) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{1.4.4}$$

Here, the Hamiltonian has the classical 'kinetic plus potential' form while (,) denotes the inner product of $\mathbb{R}^N$. Moreover, suppose that

$$\begin{aligned} &S\colon \mathbb{R}^N \to \mathbb{R}^N \;\; \text{is a symmetric linear operator with eigenvalues} \\ &\lambda_1 < 0 < \lambda_2 \le \lambda_3 \le \dots \le \lambda_N. \end{aligned}$$

This means that the quadratic form (Sp, p) is indefinite but not degenerate. Thus, the Hamiltonian system is given by

$$\begin{aligned} \dot{r}(t) &= Sp(t), \\ -\dot{p}(t) &= V'(r(t)), \end{aligned} \qquad t \in \mathbb{R}. \tag{1.4.5}$$

Hofer and Toland [65] developed a theory for the existence of homoclinic, heteroclinic and periodic orbits of (1.4.5), mainly based on topological methods like the antipodal mapping theory and the Brouwer degree theory. They considered only a class of Hamiltonian systems, for which the p-dependance is explicitly required to be *indefinite*. In [47, 113, 114, 146, 147] periodic solutions are established for a different class, by chiefly applying the theory of critical points for indefinite functionals, the study of geodesics and duality theory. Further refinements of the ideas of [65] have been made in [25], yielding the existence of a family of homoclinic orbits of (1.4.5) when the potential is of the form

$$V(u, u'') = \frac{1}{2}u^2 - \frac{1}{3}u^3 - \frac{1}{2}u''^2.$$

It is worth noting that equation (1.3.1) can be seen as an Hamiltonian system (1.4.5), where

$$r = (u, u''), \quad p = (u''' + qu', u'), \quad V(r) = F(u) - \frac{u''^2}{2}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & q \end{pmatrix},$$

and of course the Hamiltonian is given by (1.4.4). It is immediate to see that the kinetic energy is indefinite, since

$$\det \begin{pmatrix} -\lambda & 1 \\ 1 & q - \lambda \end{pmatrix} = 0 \Leftrightarrow \lambda^2 - \lambda q - 1 = 0 \Leftrightarrow \lambda_{1,2} = \frac{q \pm \sqrt{q^2 + 4}}{2},$$

which means that $\lambda_1 < 0$ for every $q \in \mathbb{R}$. The Lagrangian associated to this system is

$$\mathcal{L}(r) = \frac{1}{2}(S^{-1}r, r) - V(r).$$

Since S is symmetric but indefinite for any q, the Hamiltonian approach seems difficult to exploit. Moreover, even if (1.3.1) can be viewed in the framework of the theory developed in [65] for such indefinite systems, the potential V is by far too general for the method to directly succeed. Instead, we take advantage of the particular structure of (1.4.5) and look at it as a *higher order* Lagrangian problem. See the next section for further details.

1.4.3 Variational methods

In the study of bounded stationary solutions to equation (1.3.1) on $\mathbb{R}$, variational methods take up an important place. The reason is that (1.3.1) has a variational structure and is the Euler-Lagrange equation of the functional

$$\mathcal{J}_q(u) = \int_I L_q(u(x), u'(x), u''(x))dx, \tag{1.4.6}$$

where L_q is the second order *Lagrangian*

$$L_q(u, u', u'') := \frac{1}{2}\left(|u''|^2 - q|u'|^2\right) + F(u).$$

Recall that we are used to refer to the last term of L_q as the *potential*. Variational methods have been widely used to establish the existence of heteroclinic, homoclinic and periodic solutions as critical points of $\mathcal{J}_q$ in appropriate sets of functions (see [24, 27, 31, 68, 69]). When $q \leq 0$, $\mathcal{J}_q$ is nonnegative, so we can look for solutions of (1.3.1) as its minimizers. When $q > 0$, the functional is no more nonnegative, so the method of searching the minimizers may be no longer at hand. In any case, we can look for its critical points via other arguments, for instance by applying the Mountain Pass Theorem.

Solutions of (1.3.1) are critical points of $\mathcal{J}_q$ in different functional spaces depending on the type of solution considered. For example, for homoclinic orbits satisfying

$$\lim_{x \to \pm\infty} (u, u', u'', u''')(x) = (0, 0, 0, 0),$$

the appropriate space would be the Sobolev space $H^2(\mathbb{R})$. Chen and McKenna [31] have used this approach and employed the Mountain Pass Lemma and the Concentration Compactness Principle to prove the existence of pulses if, respectively, $f(s) = s - s^2$, as in the water wave problem, and $f(s) = (s+1)_+ - 1$, as in the suspension bridge problem. For both of these problems, they considered $-2\sqrt{f'(0)} < q < 2\sqrt{f'(0)}$, the range of values of q for which the origin is a saddle-focus.

Moreover, the integral in (1.4.6) can be taken on various sets I according to the domain of the solution and the functional space on which $\mathcal{J}_q$ is defined. For heteroclinic solutions, since they are defined on $\mathbb{R}$, we consider $I = \mathbb{R}$, leading to

$$J_q(u) = \int_{\mathbb{R}} \frac{1}{2}\left(|u''|^2 - q|u'|^2\right) + F(u)dx.$$

This functional is well defined for functions u having first and second square integrable derivatives and being such that the potential F is integrable. Taking into account conditions (1.3.5), we can then define J_q in the space

$$\{u \colon \mathbb{R} \to \mathbb{R} \,|\, u - x_1 \in H^2(\mathbb{R}^-),\, u - x_2 \in H^2(\mathbb{R}^+)\}.$$

The existence of heteroclinic solutions of (1.3.1) via variational arguments was investigated for the first time by Peletier, Troy and van der Vorst [107] and Kalies and van der Vorst [69]. For $q \leq 0$ and $F(u) = \frac{1}{4}(1 - u^2)^2$, Peletier et al. [107] proved the existence of a minimizer of J_q in the subset of odd functions of the space

$$\mathcal{E} = \{u \colon \mathbb{R} \to \mathbb{R} \,|\, u(0) = 0,\, u + 1 \in H^2(\mathbb{R}^-),\, u - 1 \in H^2(\mathbb{R}^+)\}.$$

To obtain an odd heteroclinic solution it is sufficient to look for critical points of the functional

$$J_q^+(u) := \int_{\mathbb{R}^+} \frac{1}{2}\left(|u''|^2 - q|u'|^2\right) + \frac{1}{4}(u^2 - 1)^2 dx$$

in the space

$$\mathcal{E}^+ := \{u \,|\, u - 1 \in H^2(\mathbb{R}^+),\, u(0) = 0\}.$$

Indeed, the condition $u''(0) = 0$ is a natural boundary condition fulfilled by any critical point of J_q^+ in $\mathcal{E}^+$ so that their odd extensions solve the Euler-Lagrange equation (1.3.2) on $\mathbb{R}$. These methods have been considerably refined in [68, 69]. Of course, the previous arguments extend easily to a functional defined from any second order positive Lagrangian with a symmetric potential having two non-degenerate minima (with non-vanishing second derivative) at the same energy level and superquadratic grows at $\pm\infty$. When we look for T-periodic solutions, the natural space we consider is the real Hilbert space

$$H_T := \{u : u \in H^2([0,T]),\, u' \in H^1_0([0,T])\},$$

with scalar product

$$\langle u, v\rangle_{H_T} = \int_0^T u''v''dx + \int_0^T uvdx$$

and corresponding norm $\|u\|_{H_T}$. Of course the action functional is defined by

$$J_{q,T}(u) := \int_0^T \frac{1}{2}\left(|u''|^2 - q|u'|^2\right) + F(u)dx,$$

while the dependance on T is often omitted when there is no risk of confusion.

1.5 Some results from the literature

As already said, the nature of equation (1.3.1) depends both on the choice of the potential F and on the sign of the parameter q and, of course, the existence results reflect this influence. We begin with some results involving a double-well potential, and after we will deal with a single-well coercive potential.

1.5.1 F is a double-well potential

In the model case $F(u) = \frac{1}{4}(1-u^2)^2$, equation (1.3.1), as already seen, becomes

$$u'''' + qu'' + u^3 - u = 0. \tag{1.5.1}$$

Peletier and Troy [106] gave a complete catalogue of the set $\mathcal{B}$ of bounded solutions of equation (1.5.1) for the parameter range $q \in (-\infty, -\sqrt{8}]$. Taking into account Section 1.3.1, this is the saddle-nodes case, when the spectrum at the stable uniform states $u = +1$ and $u = -1$ is real valued. Their results can be summarized as follows.

Theorem 1.5.1 (Theorems 2.1.1-2.2.1 of [106])**.** *Let $q \in (-\infty, -\sqrt{8}]$. Then, the following facts hold true:*

1. *For every $E \in (0, \frac{1}{4})$ there exists a periodic solution u_E to equation* (1.5.1)*, which is even with respect to its critical points, odd with respect to its zeros, and has the bound*
$$\|u_E\|_\infty < \sqrt{1 - 2\sqrt{E}}.$$
2. *There exists an odd monotone heteroclinic solution of equation* (1.5.1) *that satisfies*
$$\lim_{x\to\pm\infty}(u, u', u'', u''')(x) = (\pm 1, 0, 0, 0).$$

They obtained these results by using the method of topological shooting introduced in Section 1.4.1. Moreover, they proved that the solutions obtained in Theorem 1.5.1 are the *only* bounded non-constant solutions of equation (1.5.1) for these values of q.

When $q > -\sqrt{8}$, the spectrum at the equilibria $u = \pm 1$ is complex valued. An immediate consequence is that, in this parameter regime, homoclinic and heteroclinic solutions leading to either of these two constant solutions cannot be *monotone*. In fact, as q passes through $-\sqrt{8}$, the set $\mathcal{B}$ instantly becomes much richer and the solution graphs more complex (see [25, 27, 42] and the references therein). In this range, the linearization around the equilibria displays oscillatory solutions so that any heteroclinic of (1.5.1) oscillates around ± 1 in its tails, i.e., when $x \to \pm\infty$. This oscillatory behavior close to the equilibria makes a shooting method much more tedious, since one of the greatest difficulties is to control the convergence at infinity. However, Peletier and Troy adapted their arguments in [104], and after a careful analysis they managed to single out two families of odd heteroclinics in the range $q \in (-\sqrt{8}, 0]$, which differ by the amplitude of the oscillations. The first one consists of the so-called *multi-transition* solutions, since all the successive local extrema between the zeros rely outside the region $[-1, 1]$. It contains, for each $n \in \mathbb{N}$, a solution whose profile displays $2n+1$ jumps from -1 to $+1$ and two oscillatory tails around -1 and $+1$. The second family contains the *single-transition* heteroclinics, whose solutions display oscillations with an amplitude smaller than 1. They may also be classified according to their number of oscillations around 0. Both these families have a similar structure, which can be divided into three different regions: an inner region $(-L, L)$, where the solution oscillates around $u = 0$, and two outer regions, $(-\infty, -L)$ and $(L, +\infty)$, which contain the tails where the solution in the inner region joins up with one of the stable uniform states ± 1.

Moreover, Kalies and van der Vorst [69] constructed the so-called *multi-bump* solutions, which are characterized by multiple oscillations separated by large distances. The usual methods to obtain such solutions are rather tricky and require a careful study of the stable and unstable manifolds (see [27, 37, 120]). Kalies et al. [68] introduced a direct method to find multi-transition solutions. Note that such solutions are qualitatively different from multi-bump ones, as the distance between two successive transitions is not necessarily large. The method in [68] consists in minimizing the action functional J_q in specific subspaces of functions having a common homotopy type. Basically, the homotopy type describes the trajectory of any function in the uu'-plane by recording the number of transitions from one equilibrium to the other and counting the number of turns it makes around -1 and $+1$ in between the transitions. Their method perfectly handles oscillatory graphs and is therefore efficient when $-\sqrt{8} \le q \le 0$.

Existence results in the parameter range $q \le 0$ can be also obtained through different methods. For example, Peletier et al. proved the existence of heteroclinic solutions to equation (1.5.1) via variational argument, see [107].

The dynamics of equation (1.5.1) with $q > 0$ is much less understood than the EFK case. Numerical experiments (see [15]) suggest that a large variety of those solutions found for $q \le 0$ still exists for a certain range of positive values of q. However, the limitations of the shooting method of Peletier and Troy was pointed out by van den Berg [14].

As q becomes larger than $-\sqrt{8}$ (in particular positive), a multitude of periodic solutions with different structures emerges, as described in [106]. We pay specific attention to two families, that consist of odd and even periodic solutions, respectively, each with zero energy. Observe that they are only a part of the set of periodic solutions that exist in this parameter range. The first family consists in two branches of single-bump periodic solutions (with a single oscillation), which emerge at the value $q = -\sqrt{8}$. They are divided into Γ_+, with amplitude larger than one, and Γ_-, with amplitude smaller than 1. The second one consists in a countable pairs of family of branches: they are divided into Γ_{na} and Γ_{nb}, whose solutions are convex and concave at the origin, respectively. Both of these families extend over the intervals $(-\sqrt{8}, q_n)$, where

$$q_n = \sqrt{2}\left(n + \frac{1}{n}\right), \quad n = 2, 3, \ldots$$

The following is an existence result of *single-bump* periodic solutions.

Theorem 1.5.2 (Theorem 4.1.1. of [106]). *For every $q > -\sqrt{8}$ there exist single-bump periodic solutions u_- and u_+ of equation* (1.5.1) *such that $\mathcal{E}(u_\pm) = 0$ and*

$$\|u_-\|_\infty < 1 \quad \textit{and} \quad \|u_+\|_\infty > 1.$$

Moreover, the functions $u_\pm$ are odd with respect to their zeros and even with respect to their critical points.

There is also a family of odd multi-bump periodic solutions with the characteristic property that the maxima all lie above $u = 1$ and the minima all lie below $u = -1$, with the exception of the *first* point of symmetry, ζ in $\mathbb{R}^+$, where the situation is reversed (see [106, Theorem 4.1.3]).

We conclude this section with three qualitative results. The first one is a sharp universal upper bound for bounded solutions of equation (1.5.1) when $q \leq 0$, while the other two describe the asymptotic behavior of bounded solutions when $q \in (-\sqrt{8}, +\infty)$.

Theorem 1.5.3 (Lemma 2.4.3 and Lemma 2.4.5 of [106]). *If $q \leq 0$, then any bounded solution u of equation* (1.5.1) *satisfies*

$$|u(x)| < \sqrt{2}, \quad \forall\, x \in \mathbb{R}.$$

When, in particular, $q \leq -\sqrt{8}$, then

$$|u(x)| < 1, \quad \forall\, x \in \mathbb{R}.$$

For the next results, we denote by φ the odd increasing kink at $q = -\sqrt{8}$.

Theorem 1.5.4 (Theorem 4.2.2 of [106]). *Let $(q_n) \subseteq (-\sqrt{8}, +\infty)$ be a decreasing sequence such that $q_n \to -\sqrt{8}$ as $n \to +\infty$, and let $u_n(x) = u(x, q_n)$ be a corresponding sequence of odd solutions of equation* (1.5.1) *with zero energy and such that $u_n'(0) > 0$. Then,*

$$u_n \to \varphi \quad \textit{as } n \to +\infty$$

uniformly on bounded intervals.

Theorem 1.5.5 (Lemma 4.2.4 of [106]). *Let $u(x, q)$ be an odd periodic solution of equation* (1.5.1) *with zero energy, symmetric with respect to its critical points and such that $\|u(\,\cdot\,, q)\|_\infty < 1$. Then,*

$$\|u(q)\|_\infty < \frac{1}{q\sqrt{2}} \qquad \textit{for every } q > 0.$$

1.5.2 F is a single-well coercive potential

When F is a convex, coercive potential, the study of (1.3.1) became a major tool to understand the modeling of suspension bridges, see [77, 95, 96]. When we search for traveling waves that decay to zero exponentially as $|x| \to \infty$, we are ultimately concerned with finding homoclinic solutions of the nonlinear ordinary differential equation

$$y'''' + c^2 y'' + (1+y)_+ - 1 = 0. \tag{1.5.2}$$

The idea in [96] was to write the analytic expressions for the solutions of $y'''' + c^2 y'' + y = 0$ for $y \geq -1$ and $y'''' + c^2 y'' - 1 = 0$ for $y \leq -1$, and then ensure the continuity of these solutions and their first three derivatives whenever $y = -1$. However there were some problems with this approach. Indeed, the existence was not proved rigorously since all calculations were approximate. Nonetheless, the paper led to some obvious conjectures: it seemed that the number of solutions could be quite large and, moreover, the L^∞-norm of the solutions seemed to go to $+\infty$, as $c \to 0$.

Furthermore, the method of finding traveling wave solutions was heavily dependent on the analytic form of the nonlinearity in (1.5.2). Thus, the publication of this paper left open several interesting questions. First, could one verify that, as $c \to 0$, the L^∞-norm of the solutions goes to $+\infty$, as was indicated by the computations? Second, could one prove existence (and multiplicity) for solutions of a more general nonlinearity with the same basic shape as that of (1.5.2)? And third, which are the stability, instability and interaction properties of these traveling waves? In [79], Lazer and McKenna gave a result of non-existence about what happens for the parameter value $c = 0$.

Theorem 1.5.6 (Theorem 1 of [79])**.** *The solution $u \equiv 0$ is the only solution of the equation*

$$y'''' + (1+y)_+ - 1 = 0$$

such that $\|u\|_\infty$ is bounded.

In order to give to the reader the idea of what it is known so far in the literature, below we list some results about the argument. Note that the hypotheses on F have been further specialized, leading to consider coercive and quasi-convex potentials, i.e., those satisfying

$$F'(t)t \geq 0, \qquad \forall\, t \in \mathbb{R}. \tag{1.5.3}$$

Existence results

Let us first remark that, for coercive potentials, condition (1.5.3) has proven to be almost equivalent to the absence of (nontrivial) bounded solutions to (1.3.1). Here there is a more precise statement.

Theorem 1.5.7 (Theorems 3.1 and 5.1 in [97])**.** *Let $q \leq 0$. If* (1.5.3) *holds, then the only bounded solutions of* (1.3.1) *are constants. Moreover, in the class of coercive potentials with $\{F'(t) = 0\}$ discrete,* (1.5.3) *is actually* equivalent *to the existence of bounded nontrivial solutions.*

Under assumption (1.5.3), one may still seek for global unbounded solutions. An immediate ODE argument shows that, if F' is globally Lipschitz, all local solutions of (1.3.1) are actually global (and thus, by the previous theorem, unbounded).

However, the peculiar nature of (1.3.1) allows the following *one-sided* generalization (notice that this holds *for any* $q \in \mathbb{R}$).

Theorem 1.5.8 (Theorem 1 in [13]). *Let $q \in \mathbb{R}$ be arbitrary and $F \in C^2$ satisfy $F'(t)t > 0$, for all $t \neq 0$. If*

$$\text{either} \quad \limsup_{t\to+\infty} \frac{F'(t)}{t} < +\infty \quad \text{or} \quad \limsup_{t\to-\infty} \frac{F'(t)}{t} < +\infty, \tag{1.5.4}$$

then any solution to (1.3.1) *is globally defined.*

Regarding non-existence, Gazzola and Karageorgis proved the following. Recall that with $\mathcal{E}$ we mean the Hamiltonian energy (1.3.3).

Theorem 1.5.9 (Theorem 3 in [53]). *Let $q \leq 0$. Suppose F is a convex potential satisfying*

$$F(0) = 0, \qquad F'(t)t \geq c|t|^{2+\varepsilon} \quad \text{for } \varepsilon > 0, \qquad F'(t)t \geq cF(t) \quad \forall\, |t| >> 1$$

for some $c > 0$ and

$$\liminf_{|t|\to+\infty} \frac{F(\lambda t)}{F(t)^\alpha} > 0$$

for some $\lambda \in]0, 1[$, $\alpha > 0$. If u solves (1.3.1) *in a neighborhood of* 0 *and*

$$\text{either} \quad u'(0)u''(0) - u(0)u'''(0) - qu(0)u'(0) \neq 0 \qquad \text{or} \quad \mathcal{E} \neq 0, \tag{1.5.5}$$

then u blows up in finite time.

As we will see, the situation for $q > 0$ is more complex. Regarding non-existence of nontrivial solutions, the seemingly most up-to date results are the following.

Theorem 1.5.10 (Theorem 1 in [115]). *Let $F \in C^2$ satisfy*

$$a|t|^{p+1} \leq F'(t)t \leq b|t|^{r+1} + c|t|^{p+1}, \quad \text{for some } a, b, c > 0 \text{ and } 1 \leq r < p\,. \tag{1.5.6}$$

Then, for any $q > 0$[2]*, there exists $E_0 = E_0(a, b, c, p, r, q) \geq 0$ such that any solution to* (1.3.1)*, satisfying $\mathcal{E}(u) > E_0$, blows up in finite time.*

Theorem 1.5.11 (Theorem 1 in [49]). *Let $F \in C^2$ satisfy* (1.5.6) *and*

$$F''(t) > F''(0), \quad \text{for all } t \neq 0. \tag{1.5.7}$$

If $q > 0$ satisfies $q^2 \leq 4F''(0)$, then the only globally defined solution to (1.3.1) *is $u \equiv 0$.*

Still in [49], the rôle of the condition $q < 2\sqrt{F''(0)}$ is also discussed, through the following partial converse of Theorem 1.5.11.

Theorem 1.5.12 (Theorem 2 in [49]). *Suppose $F \in C^2$ is even, satisfies* (1.5.6), (1.5.7) *and the limit $\lim_{t\to+\infty} \frac{F'(t)}{t^p}$ exists. Then, for every $q > 0$ such that $q^2 > 4F''(0)$, there exists a nontrivial periodic solution to* (1.3.1).

Notice that, being $p > 1$, (1.5.6) forces

$$\lim_{t\to+\infty} \frac{F'(t)}{t^p} = B > 0,$$

which implies the (much weaker) condition

$$\liminf_{|t|\to+\infty} \frac{F(t)}{t^2} = +\infty.$$

[2]In [115], this theorem is actually proved for $0 < q \leq 2$, but a simple scaling argument shows its validity for any $q > 0$. Indeed, if u solves (1.3.1), then $u_\lambda(x) := u(x/\sqrt{\lambda})$ solves $u_\lambda'''' + \lambda q u_\lambda'' + F_\lambda'(u_\lambda) = 0$, where $F_\lambda(t) := \lambda^2 F(t)$ satisfies (1.5.6) with the same exponents and with constants a, b, c multiplied by λ^2. If $\lambda \leq 2/q$, one can then apply [115, Theorem 1].

Asymptotic behavior

In light of the previous discussion, under assumption (1.5.3) it only makes sense to consider the asymptotic behavior of the solutions to (1.3.1) for $q \downarrow 0$. The starting point is a result proved by Lazer and McKenna.

Theorem 1.5.13 (Theorem 2 in [79])**.** *Let, for some $q_n \downarrow 0$, (u_n) be a sequence of bounded nontrivial solutions to*

$$u'''' + q_n u'' + (1+u)_+ - 1 = 0.$$

Then, $\|u_n\|_\infty \to +\infty$.

We briefly say that *the nontrivial solutions to* (1.3.1) *are unbounded as $q \downarrow 0$* if the thesis of the previous theorem holds for any $q_n \downarrow 0$ and corresponding nontrivial solutions (u_n) to (1.3.1). The previous result has later been generalized as follows.

Theorem 1.5.14 (Theorem 3.2 in [97])**.** *Let $F \in C^2$ satisfy $F''(0) > 0$, $\mathrm{int}(\{F' = 0\}) = \emptyset$ and* (1.5.3)*. Then, the nontrivial solutions to* (1.3.1) *are unbounded, as $q \downarrow 0$.*

The condition $\mathrm{int}(\{F' = 0\}) = \emptyset$ is readily seen to be necessary for the thesis, as the following remark shows.

Remark 1.5.1 (Remark 3.2 in [97])**.** *Suppose that $[a,b] \subseteq \{F' = 0\}$, then $u_\beta(x) := A\sin(\beta x) + B$ with $A = (b-a)/4, B = (a+b)/2$ solves* (1.3.1) *for each β, being uniformly bounded.*

1.6 Our results

In [92] we gave some answers to the questions raised by Lazer and McKenna in [79], considering equation (1.3.1) for coercive, quasi-convex potentials F. For the reader's convenience, below we recall that (1.3.1) reads as

$$u'''' + qu'' + F'(u) = 0. \tag{1.6.1}$$

Notice that some of our statements will involve quantities like

$$\limsup_{t\to\pm\infty} \frac{F(t)}{t^2}, \qquad \liminf_{t\to\pm\infty} \frac{F(t)}{t^2}.$$

Similar, but weaker, statements will hold under similar conditions on $f(t) = F'(t)$, namely involving the corresponding quantites

$$\limsup_{t\to\pm\infty} \frac{f(t)}{t}, \qquad \liminf_{t\to\pm\infty} \frac{f(t)}{t}$$

(which are more frequent in the literature), simply due to the inequalities

$$\limsup_{t\to\pm\infty} \frac{F(t)}{t^2} \le \limsup_{t\to\pm\infty} \frac{f(t)}{t}, \qquad \liminf_{t\to\pm\infty} \frac{F(t)}{t^2} \ge \liminf_{t\to\pm\infty} \frac{f(t)}{t},$$

which may be strict in some cases.

Since we are looking for periodic solutions, (1.6.1) could be seen as the Euler-Lagrange equation of the functional

$$J_q(u) := \int_0^T \frac{1}{2}\left(|u''|^2 - q|u'|^2\right) + F(u)dx,$$

defined on the Hilbert space

$$H_T := \{u : \, u \in H^2([0,T]), \, u' \in H_0^1([0,T])\}.$$

(see Section 1.4.3 for further details). Observe that we can freely add to F and J_q a constant so that $J_q(0) = F(0) = 0$. Since we are mainly interested in potentials satisfying (1.5.3), this implies $F(0) = \min_{t\in\mathbb{R}} F(t)$. Thus, we can reduce to the case

$$0 = F(0) = \min_{t\in\mathbb{R}} F(t), \tag{1.6.2}$$

a weaker hypothesis we will sometimes assume. The link between solutions of (1.6.1) and the functional J_q is given in the following proposition.

Proposition 1.6.1 (Lemma 4.1 in [97]). *Let $u\colon [0,T] \to \mathbb{R}$ be a critical point for J_q in H_T. Then, its even extension $\tilde{u}\colon [-T,T] \to \mathbb{R}$ defines a $2T$-periodic $C^4(\mathbb{R})$ solution to* (1.6.1)*.*

Proof. Integrating by parts and using a local uniqueness for the ODE, we see that any critical point $u \in H_T$ of J_q satisfies (1.6.1) and belongs to $C^4([0,T])$. Since $u \in H_T$ forces $u'(0) = u'(T) = 0$, it suffices to show that $u'''(0) = u'''(T) = 0$. Integrating by parts $(DJ_q(u), \phi) = 0$ for $\phi \in H_T$ (and thus $\phi'(0) = \phi'(T) = 0$), we get

$$\begin{aligned} 0 &= \int_0^T u''\phi'' - qu'\phi' + F'(u)\phi dx \\ &= -\int_0^T u'''\phi' - qu''\phi - F'(u)\phi dx = -u'''\phi|_0^T, \end{aligned}$$

being $u \in H_T$ a solution of (1.6.1). Clearly we can choose arbitrarily the values 0 and T for $\phi \in H_T$, obtaining $u'''(0) = u'''(T) = 0$. Thus the even extension of u belongs to $C^4(\mathbb{R})$ and by local uniqueness, it is a $2T$-periodic solution of (1.6.1). □

We will furthermore use the following elementary observation.

Lemma 1.6.1. *For any $T > 0$, $J_q\colon H_T \to \mathbb{R}$ is C^1 and weakly sequentially lower semi-continuous.*

Proof. The functional J_q being C^1 immediately follows from the Sobolev embedding

$$\|u\|_\infty \le C_T \|u\|_{H_T},$$

which ensures that there is no need of a growth condition for F. To prove lower semi-continuity, let $(v_n) \subseteq H_T$ be a sequence such that $v_n \rightharpoonup v$, weakly in H_T. In particular, (v_n) is bounded in H_T, which implies boundedness in $C^{1,\alpha}([0,T])$, by Sobolev embedding. Therefore, (v_n) is compact in $C^1([0,T])$ by Ascoli-Arzelà, which ensures, by Lebesgue dominated convergence, that

$$-q\int_0^T |v_n'|^2\, dx + \int_0^T F(v_n)\, dx \to -q\int_0^T |v'|^2\, dx + \int_0^T F(v)\, dx.$$

Since the remaining term $\frac{1}{2}\|v_n''\|_2^2$ is weakly sequentially lower semi-continuous by convexity, the thesis follows. □

We conclude with a general result on convex envelopes, which is of some interest in itself. More precisely, given $G\colon \mathbb{R}^N \to \mathbb{R}$, we let

$$G^*(x) := \sup\Big\{g(x) : g \text{ is convex and } g(y) \le G(y), \text{ for all } y \in \mathbb{R}^N\Big\}$$

be the convex envelope of G.

Lemma 1.6.2. *Suppose $G\colon \mathbb{R}^N \to \mathbb{R}$ is a lower semi-continuous function such that*

$$\liminf_{|x|\to+\infty} \frac{G(x)}{|x|} \ge \alpha, \quad \textit{for some } \alpha \in \,]0,+\infty[. \tag{1.6.3}$$

Then,

$$\mathrm{Argmin}(G^*) = \mathrm{co}\big(\mathrm{Argmin}(G)\big). \tag{1.6.4}$$

Proof. Clearly, $\mathrm{Argmin}(G)$, and thus $\mathrm{co}\big(\mathrm{Argmin}(G)\big)$, is compact and non empty, and we can suppose, without loss of generality, that $\min_{\mathbb{R}^N} G = 0$. From (1.6.3) we can find $M > 0$ such that $G(x) \ge \frac{\alpha}{2}|x|$, for any $|x| \ge M$, hence the function

$$h(x) = \frac{\alpha}{2}(|x| - M)$$

satisfies $h \le G$ in $\mathbb{R}^N$. Since $G^* \ge h$ by construction, this implies that $\mathrm{Argmin}(G^*)$ is compact as well.

Since $g \equiv 0$ satisfies $g \le G$ and is convex, it holds $0 \le G^* \le G$, which implies that $\mathrm{co}\big(\mathrm{Argmin}(G)\big) \subseteq \mathrm{Argmin}(G^*)$. We prove the opposite inequality by contradiction, and thus suppose that there exists x_0 such that

$$G^*(x_0) = 0 \text{ and } x_0 \notin \mathrm{co}\big(\mathrm{Argmin}(G)\big) =: C. \tag{1.6.5}$$

By the Hanh-Banach Theorem, there exists $v \in \mathbb{R}^N$, $|v| = 1$, such that

$$\sup_{x\in C}\langle v, x\rangle = \langle v, x_1\rangle < \langle v, x_0\rangle, \quad \text{for some } x_1 \in C, \tag{1.6.6}$$

where, by $\langle v, x\rangle$, we mean the standard duality coupling. Let, for $\varepsilon > 0$

$$g_\varepsilon(x) = \varepsilon\langle v, x - \frac{x_0 + x_1}{2}\rangle,$$

and notice that for any $x \in C$ it holds, by (1.6.6),

$$g_\varepsilon(x) \le \varepsilon\left(\sup_{y\in C}\langle v, y\rangle - \langle v, \frac{x_0 + x_1}{2}\rangle\right) = \varepsilon\left(\langle v, x_1\rangle - \langle v, \frac{x_0 + x_1}{2}\rangle\right) = \varepsilon\langle v, \frac{x_1 - x_0}{2}\rangle < 0.$$

Therefore, for any $\varepsilon > 0$,

$$\sup_{x\in C} g_\varepsilon(x) < 0. \tag{1.6.7}$$

The set $\{g_{\alpha/4} \ge h\}$ is compact, since

$$\limsup_{|x|\to+\infty} \frac{g_{\alpha/4}(x)}{h(x)} = \limsup_{|x|\to+\infty} \frac{1}{2}\frac{\langle v, x - \frac{x_0+x_1}{2}\rangle}{|x| - M} = \frac{1}{2}.$$

Thus, $K := \{g_{\alpha/4} \ge 0\} \cap \{g_{\alpha/4} \ge h\}$ is compact and, for any $x \in K$, it holds $G(x) > 0$ since, otherwise, $x \in C$ and (1.6.7) implies $g_{\alpha/4}(x) < 0$, contradicting $x \in K$. Therefore, we can set

$$\inf_{x\in K} G(x) = \beta > 0.$$

Now, for sufficiently small $\varepsilon \in]0, \alpha/4[$, it holds

$$\sup_{x \in K} g_\varepsilon(x) \le \beta,$$

and we claim that for such ε it holds $g_\varepsilon \le G$ in the whole $\mathbb{R}^N$. This is clearly true on $\{g_\varepsilon \le 0\} = \{g_{\alpha/4} \le 0\}$, since $G \ge 0$. From the definition of ε it holds

$$g_\varepsilon(x) \le \beta \le G(x), \quad \text{for all } x \in K.$$

Finally, on $\{g_{\alpha/4} \ge 0\} \cap \{g_{\alpha/4} < h\}$ one has

$$g_\varepsilon(x) \le g_{\alpha/4}(x) < h(x) \le G(x),$$

and the claim is proved. Therefore, being g_ε convex, we deduce $G^* \ge g_\varepsilon$. By (1.6.6),

$$G(x_0) \ge g_\varepsilon(x_0) = \varepsilon \langle v, \frac{x_0 - x_1}{2} \rangle > 0,$$

which gives the desired contradiction to (1.6.5). □

Remark 1.6.1. *Condition* (1.6.3) *is optimal in order to obtain* (1.6.4)*. Consider the* $C^2(\mathbb{R}, \mathbb{R})$ *coercive function*

$$G(x) = \begin{cases} \log(1 + x^2) & \text{if } x < 0, \\ x^2 & \text{if } x \ge 0. \end{cases}$$

A straightforward computation shows that

$$G^*(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^2 & \text{if } x \ge 0, \end{cases}$$

and (1.6.4) *fails, since* $\mathrm{Argmin}(G^*) =]-\infty, 0] \ne \{0\} = \mathrm{co}(\mathrm{Argmin}(G))$.

We obtained several existence results both for the EFK equation ((1.6.1) when $q \le 0$) and for the S-H equation ((1.6.1) when $q > 0$) and we list them below, separately. At the end of the section, we will also describe some results regarding the asymptotic behavior of solutions.

1.6.1 The EFK case

We begin with a result which can be seen as an immediate consequence of Theorem 1.5.9.

Theorem 1.6.1. *Let* $q \le 0$*. Suppose* F *satisfies the assumptions of Theorem 1.5.9. Then, the only globally defined solution to* (1.6.1) *is* $u \equiv 0$.

Proof. By translation invariance, $u(x + x_0)$ is a global solution to (1.6.1), for any $x_0 \in \mathbb{R}$. Therefore, the first condition in (1.5.5) must fail at any $x_0 \in \mathbb{R}$, which implies that

$$u'u'' - uu''' - quu' \equiv 0.$$

Deriving this relation, we obtain

$$|u''|^2 - u(u'''' + qu'') - q|u'|^2 = |u''|^2 - q|u'|^2 + F'(u)u \equiv 0$$

and, being $F'(t)t \ge c|t|^{2+\varepsilon}$ and $q \le 0$, we immediately deduce $u \equiv 0$. □

The next result is a useful tool for our further calculations. First, we recall some elementary interpolation inequalities, which provide bounds for higher derivatives of any solution u to (1.6.1) in terms of $\|u\|_\infty$. For any *unbounded* interval I, integers $0 \le j \le k, i \ge 0$ and $u \in C^{k+i}(I)$, it holds

$$\|u^{(k)}\|_{L^\infty(I)} \le c_{k,i,j} \|u^{(k+i)}\|_{L^\infty(I)}^{\frac{j}{i+j}} \|u^{(k-j)}\|_{L^\infty(I)}^{\frac{i}{i+j}}, \tag{1.6.8}$$

for a constant $c_{k,i,j}$ independent of I.

Theorem 1.6.2 (Theorem 3.1 of [97]). *Let $F \in C^1(\mathbb{R})$ be a coercive potential with a unique local (and thus global) minimum. Then, every bounded solution to* (1.6.1) *is constant.*

Proof. Let $\min_{\mathbb{R}} F = F(t_0)$ and u be a bounded solution of (1.6.1). Eventually considering $G(t) = F(t - t_0) - F(t_0)$ and $v = u + t_0$, we can suppose that 0 is the unique global minimum of F and $F(0) = 0$. In particular it holds $tF'(t) \ge 0$ for all $t \in \mathbb{R}$. Using (1.6.1), the interpolation inequalities (1.6.8) for $I = \mathbb{R}$ and Young's inequality $2ab \le a^2 + b^2$, we get

$$\begin{aligned}
\|u''''\|_\infty &\le q\|u''\|_\infty + \sup_{[-\|u\|_\infty, \|u\|_\infty]} |F'(u)| \le qc\|u\|_\infty^{1/2}\|u''''\|_\infty^{1/2} + \sup_{[-\|u\|_\infty, \|u\|_\infty]} |F'(u)| \\
&\le \frac{1}{2}\|u''''\|_\infty + \frac{q^2c^2}{2}\|u\|_\infty + \sup_{[-\|u\|_\infty, \|u\|_\infty]} |F'(u)|,
\end{aligned}$$

and thus u'''' is uniformly bounded. Now (1.6.8) for $I = \mathbb{R}$ implies that all the lower order derivatives are bounded. Consider the auxiliary function

$$h = u''u + \frac{q}{2}u^2 - u'^2.$$

By the previous discussion we have that h is a bounded function, and a straightforward calculation shows that

$$h'' = (u'''' + qu'')u - u''^2 + qu'^2 = -F'(u)u - u''^2 + qu'^2,$$

where we used (1.6.1) in the last equality. Since $F'(t)t \ge 0$ for every $t \in \mathbb{R}$ and $q \le 0$, we get that h is a bounded concave function, thus $h \equiv k \in \mathbb{R}$. This in turn implies

$$0 = h'' = -F'(u)u - u''^2 + qu'^2 \le -u''^2 \Rightarrow u''^2 \le 0,$$

and therefore u is affine. Since it is bounded, it must be constant. $\square$

Using a classical technique essentially due to Bernis [16], we remove most of the assumptions of Theorem 1.6.1, proving the following result.

Theorem 1.6.3. *Let $F \in C^2$ satisfy* (1.5.3) *and*

$$\liminf_{|t|\to\infty} \frac{F'(t)}{t|t|^\varepsilon} > 0, \qquad \varepsilon > 0. \tag{1.6.9}$$

Then, the only globally defined solutions of (1.6.1) *for $q \le 0$ are constants.*

Proof. We let, for simplicity, $p = -q \ge 0$ and $F'(t) = f(t)$. The weak formulation of (1.6.1) is

$$\int u''\varphi'' + pu'\varphi' + f(u)\varphi \, dx = 0, \qquad \text{for all} \quad \varphi \in C_c^2(\mathbb{R}).$$

Letting $\varphi = u\eta$, we have

$$\int |u''|^2\eta + 2u''u'\eta' + u''u\eta'' + p|u'|^2\eta + quu'\eta' + f(u)u\eta\, dx = 0$$

and, by Young's inequality,

$$\int |u''|^2\eta+p|u'|^2\eta+f(u)u\eta\, dx \le \int \frac{1}{2}\eta|u''|^2+\frac{1}{2}\frac{|\eta''|^2}{\eta}u^2+\frac{p}{2}\eta|u'|^2+\frac{p}{2}\frac{|\eta'|^2}{\eta}u^2\, dx-2\int u''u'\eta'\, dx.$$

It follows that

$$\int |u''|^2\eta + p|u'|^2\eta + 2f(u)u\eta\, dx \le \int u^2\left(\frac{|\eta''|^2}{\eta} + p\frac{|\eta'|^2}{\eta}\right)\, dx - 4\int u''u'\eta'\, dx. \tag{1.6.10}$$

We estimate the last term integrating by parts as

$$\int u''u'\eta'\, dx = \int \left(\frac{|u'|^2}{2}\right)'\eta'\, dx = -\int \frac{|u'|^2}{2}\eta''\, dx = -\frac{1}{2}\int u'(u'\eta'')\, dx = \frac{1}{2}\int uu''\eta''+uu'\eta'''\, dx.$$

Moreover,

$$\int uu'\eta''' dx = \int \left(\frac{u^2}{2}\right)'\eta''' dx = -\int \frac{u^2}{2}\eta'''' dx$$

and, again by Young's inequality,

$$\int uu''\eta'' dx \le \int \frac{1}{2}\eta|u''|^2 + \frac{1}{2}\frac{|\eta''|^2}{\eta}u^2 dx.$$

Inserting into (1.6.10), and using $f(t)t \ge 0$ for all $t \in \mathbb{R}$, we obtain

$$\int |u''|^2\eta + p|u'|^2\eta + f(u)u\eta\, dx \le C\int u^2\Big(\frac{|\eta''|^2 + p|\eta'|^2}{\eta} + |\eta''''|\Big)\, dx. \tag{1.6.11}$$

Fix $m \in \mathbb{N}$ large, $R > 1$, and let $\eta = \varphi_R^m$, where $\varphi_R(x) = \varphi(\frac{x}{R})$ and $\varphi \in C_c^\infty(\mathbb{R}, [0, 1])$ is a nonnegative cut-off function such that

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{if } |x| \ge 2. \end{cases}$$

Using $0 \le \varphi_R \le 1$ and $|\varphi_R^{(i)}| \le C/R^i$, an explicit calculation shows that

$$\frac{|\eta'|^2}{\eta} \le \frac{C}{R^2}\varphi_R^{m-2} \le \frac{C}{R^2}\varphi_R^{m-4}, \qquad \frac{|\eta''|^2}{\eta} \le \frac{C}{R^4}\varphi_R^{m-4}, \qquad |\eta''''| \le \frac{C}{R^4}\varphi_R^{m-4}.$$

With this choice, (1.6.11) implies, through Hölder's inequality, $R > 1$ and for any $m > 4\frac{r}{r-2}$, $r > 2$, that

$$\begin{aligned}
\int f(u)u\varphi_R^m\, dx &\le \frac{C}{R^2}\int u^2\varphi_R^{m-4}\, dx \\
&\le \frac{C}{R^2}\int u^2\varphi_R^{\frac{2m}{r}}\varphi_R^{(1-\frac{2}{r})m-4}\, dx \\
&\le \frac{C}{R^2}\left(\int |u|^r\varphi_R^m\, dx\right)^{\frac{2}{r}}\left(\int \varphi_R^{m-4\frac{r}{r-2}}\, dx\right)^{1-\frac{2}{r}} \\
&\le \frac{C}{R^2}\left(\int |u|^r\varphi_R^m\, dx\right)^{\frac{2}{r}} R^{1-\frac{2}{r}},
\end{aligned}$$

where we used that $\operatorname{supp}(\varphi_R) \subseteq [-2R, 2R]$. Observe that (1.6.9) implies that there exist $K > 0$, $\delta > 0$ such that $f(t)t > \delta|t|^{2+\varepsilon}$, when $|t| > K$. Letting $r = 2 + \varepsilon$ and choosing $m > 4\frac{2+\varepsilon}{\varepsilon}$, it follows, by Young's inequality, that

$$\begin{aligned}\delta \int_{\{|u|>K\}} |u|^r \varphi_R^m \, dx &\leq \int_{\{|u|>K\}} f(u) u \varphi_R^m \, dx \leq \frac{C}{R^{1+\frac{r}{2}}} \left(\int |u|^r \varphi_R^m \, dx \right)^{\frac{2}{r}} \\ &\leq \frac{C}{R^{1+\frac{2}{r}}} \left[\left(\int_{\{|u|>K\}} |u|^r \varphi_R^m \, dx \right)^{\frac{2}{r}} + \left(\int_{\{|u|\leq K\}} |u|^r \varphi_R^m \, dx \right)^{\frac{2}{r}} \right] \\ &\leq \frac{\delta}{2} \int_{\{|u|>K\}} |u|^r \varphi_R^m \, dx + \frac{C_{\delta,r}}{(R^{1+\frac{2}{r}})^{\frac{r}{r-2}}} + \frac{CK^2}{R}.\end{aligned}$$

Absorbing to the left the first term on the right, we obtain

$$\int_{\{|u|>K\}} |u|^r \varphi_R^m \, dx \leq \frac{C_{\delta,r}}{R^{\frac{r+2}{r-2}}} + \frac{CK^2}{R} \to 0, \quad \text{for } R \to +\infty,$$

which implies that $\|u\|_\infty \leq K$. Therefore, by Theorem 1.6.2, u is constant. $\square$

1.6.2 The S-H case

In order to investigate the optimality of the hypotheses in Theorem 1.5.11, we prove the following existence result. Notice that the main assumption is one-sided (much in the spirit of (1.5.4)) and can be required to hold at $+\infty$ instead.

Theorem 1.6.4. *Let $F \in C^2$ satisfy* (1.6.2). *For almost every $q > 0$ such that*

$$24 \limsup_{t\to-\infty} \frac{F(t)}{t^2} < q^2 < 4F''(0), \tag{1.6.12}$$

there exists a nontrivial periodic solution to (1.6.1).

We will divide the proof in some lemmas, letting in the following

$$\alpha := \limsup_{t\to-\infty} \frac{F(t)}{t^2}.$$

Lemma 1.6.3. *Let F satisfy* (1.6.2). *For any $0 \leq b < 2\sqrt{F''(0)}$ and any $T > 0$, there exist $\varepsilon > 0$ and $\theta > 0$ such that*

$$J_q(u) \geq \theta \|u\|_{H_T}^2, \qquad \text{for all } q \leq b \text{ and } \|u\|_{H_T} \leq \varepsilon.$$

Proof. We will suppose that $F''(0) > 0$, otherwise there is nothing to prove. We choose $\eta \in \,]0,1[$ such that $b < 2\sqrt{\eta F''(0)}$. For a sufficiently small M it holds, by Taylor's formula,

$$F(t) \geq \eta F''(0) \frac{t^2}{2}, \quad \text{for all } |t| \leq M.$$

Since

$$\|u\|_\infty \leq C_T \|u\|_{H_T},$$

we have that, for $\varepsilon := M/C_T$, it holds

$$\int_0^T F(u) \, dx \geq \eta F''(0) \int_0^T \frac{u^2}{2} \, dx, \quad \text{for all } \|u\|_{H_T} \leq \varepsilon. \tag{1.6.13}$$

Integrating by parts, using $u \in H_T$ and applying Hölder's inequality, we get

$$\int_0^T |u'|^2 dx = [uu']_0^T - \int_0^T uu'' dx \le \left(\int_0^T u^2 dx\right)^{1/2} \left(\int_0^T |u''|^2 dx\right)^{1/2}. \qquad (1.6.14)$$

By Young's inequality in the form $2ab \le \lambda a^2 + b^2/(4\lambda)$, we obtain

$$\int_0^T |u'|^2 \le \lambda \int_0^T |u''|^2 \, dx + \frac{1}{4\lambda} \int_0^T u^2 \, dx,$$

which we rewrite as

$$b \int_0^T \frac{|u'|^2}{2} \le \lambda b \int_0^T \frac{|u''|^2}{2} \, dx + \left(\frac{b}{4\lambda} - \eta F''(0)\right) \int_0^T u^2 \, dx + \eta F''(0) \int_0^T \frac{u^2}{2} \, dx.$$

Using (1.6.13) we thus have, for all $\|u\|_{H_T} \le \varepsilon$,

$$b \int_0^T \frac{|u'|^2}{2} \le \lambda b \int_0^T \frac{|u''|^2}{2} \, dx + \int_0^T F(u) \, dx + \left(\frac{b}{4\lambda} - \eta F''(0)\right) \int_0^T u^2 \, dx.$$

Rearranging and using $J_q(u) \ge J_b(u)$, we obtain

$$J_q(u) \ge (1 - \lambda b) \int_0^T |u''|^2 \, dx + \left(\eta F''(0) - \frac{b}{4\lambda}\right) \int_0^T u^2 \, dx,$$

if $\|u\|_{H_T} \le \varepsilon$. Since $b^2 < 4\eta F''(0)$ by assumption, we can choose

$$\bar\lambda \in \left] \frac{b}{4\eta F''(0)}, \frac{1}{b} \right[,$$

obtaining the claim with

$$\theta = \min\left\{1 - \bar\lambda b, \eta F''(0) - \frac{1}{4\bar\lambda}\right\} > 0.$$

□

Lemma 1.6.4. *For any $a > 0$ such that $a^2 > 24\alpha$ and all $T > 0$ such that*

$$\frac{a - \sqrt{a^2 - 24\alpha}}{2} < \frac{\pi^2}{T^2} < \frac{a + \sqrt{a^2 - 24\alpha}}{2}, \qquad (1.6.15)$$

there exists a function $\tilde u \in H_T$ such that $J_q(\tilde u) < 0$, for all $q \ge a$.

Proof. Fix $\theta > 1$ and let, for $\mu \ge 2$, $u_\mu(x) := \mu\big(\cos(\frac{\pi}{T}x) - \theta\big) \le (1-\theta)\mu$. An explicit calculation shows that

$$\int_0^T |u''|^2 dx = \frac{\pi^4}{2T^3}, \qquad \int_0^T |u'|^2 dx = \frac{\pi^2}{2T}, \qquad (1.6.16)$$

so that

$$J_q(u_\mu) = \mu^2 \frac{T}{4} \left(\frac{\pi}{T}\right)^2 \left[\left(\frac{\pi}{T}\right)^2 - q\right] + \int_0^T F(u_\mu) dx.$$

Furthermore, for any $\varepsilon > 0$ there exists $K > 0$ such that

$$F(t) \le (\alpha + \varepsilon) t^2, \qquad \text{for all} \quad t \le -K.$$

Letting now $\mu \ge K/(\theta - 1)$, we deduce that $u_\mu \le (1 - \theta)\mu \le -K$ and thus

$$\int_0^T F(u_\mu) \, dx \le (\alpha + \varepsilon)\mu^2 \int_0^T \left(\cos(\frac{\pi}{T}x) - \theta\right)^2 dx \le (\alpha + \varepsilon)\mu^2 \left(\frac{T}{2} + \theta^2 T\right).$$

Since $q \geq a$, we infer

$$J_q(u_\mu) \leq \mu^2 \frac{T}{4}\left[\left(\frac{\pi}{T}\right)^4 - a\left(\frac{\pi}{T}\right)^2 + 2(\alpha+\varepsilon)(1+2\theta^2)\right]$$

and, letting $z = \pi^2/T^2$, $J_q(u_\mu) < 0$ amounts to the existence of positive solutions to $z^2 - az + 2(\alpha+\varepsilon)(1+2\theta^2) < 0$, i.e.

$$a^2 < 8(\alpha+\varepsilon)(1+2\theta^2).$$

Letting $\varepsilon \to 0$ and $\theta \to 1$, we obtain the claim. $\square$

Thanks to Lemmas 1.6.3 and 1.6.4, we obtain that J_q has the so-called *mountain pass geometry*, uniformly on compact subintervals $[a,b] \subseteq]\sqrt{24\alpha}, 2\sqrt{F''(0)}[$. Indeed, it suffices to observe that $T = \sqrt{2}\pi/\sqrt{a}$ satisfies (1.6.15) and consider J_q on H_T. Moreover, $q \mapsto J_q(u)$ is monotone non-increasing in q, for any $u \in H_T$, and thus Struwe's monotonicity trick provides, for a.e. $q \in [a,b]$, a bounded Palais-Smale sequence $(u_n) \subseteq H_T$ for J_q, at the mountain-pass level $c_{q,T} > 0$. Reasoning as in [123], this in turn provides a mountain-pass critical point $u_q \in H_T$, that is, through Proposition 1.6.1, a $2T$-periodic solution of (1.6.1). The proof of Theorem 1.6.4 is thus complete.

Remark 1.6.2. *Inspecting the proof, one can rephrase the previous theorem putting more emphasis on the possible periods for which existence occurs. Namely, since we are assuming* $6\alpha < F''(0)$ *(otherwise there is nothing to prove), elementary calculus shows that then*

$$\inf_{z>0} z + \frac{6\alpha}{z} < 2\sqrt{F''(0)}.$$

Therefore, we can fix $T > 0$ *such that*

$$\frac{\pi^2}{T^2} + \frac{6\alpha T^2}{\pi^2} < 2\sqrt{F''(0)}.$$

Then the previous proof shows that, for almost any q *such that*

$$\frac{\pi^2}{T^2} + \frac{6\alpha T^2}{\pi^2} < q < 2\sqrt{F''(0)},$$

there exists a $2T$*-periodic solution to* (1.6.1)*.*

The numbers that appear in (1.6.12) are probably not optimal, however they allow the construction of an example showing that condition (1.5.7) is *essential* for non-existence.

Example 1.6.1. *We claim that there exists* $F \in C^2$ *such that* $F''(0) > 0$ *and* (1.5.6) *holds, with* (1.6.1) *having a nontrivial periodic (thus* bounded*) solution, for almost every* $q > 0$ *such that* $q^2 < 4F''(0)$*.*

Indeed, one can easily construct $F \in C^2$ *such that*

1. $F''(0) > 0 = F(0)$,
2. $F'(t)t > 0$, *for all* $t \neq 0$,
3. *it holds*

$$\limsup_{t \to -\infty} \frac{F(t)}{t^2} = 0.$$

Then, Theorem 1.6.4 provides a periodic solution u for almost every $q \in [0, 2\sqrt{F''(0)}]$. *Using Taylor's formula,* (1.5.6) *holds with* $1 = r < p = 2$ *and some* $a, b > 0$, *for* t *in a* bounded *open neighborhood* U *of* $\overline{u(\mathbb{R})}$. *Moreover, we can modify* F *outside of* $\overline{u(\mathbb{R})}$ *so that* (1.5.6) *holds anywhere.*

The direct method of Calculus of Variations immediately provides the following existence result, Theorem 1.6.5, which allows us to generalize Theorem 1.5.12. First, we put two lemmas that are useful tools for the proof of the theorem. For a precise range of the values of T for which the thesis holds, we refer to the remark below.

Lemma 1.6.5. *Let* $F \in C^2$ *with* $F(0) = F'(0) = 0$. *For any* $q > 2\sqrt{F''(0)}$, *there exists* $T > 0$ *such that* $\inf_{H_T} J_q < 0$.

Proof. Choosing $u(x) = \cos(\frac{\pi}{T}x)$, we have $\int_0^T u^2\,dx = \frac{T}{2}$. For any $\varepsilon > 0$ there exists $M > 0$ such that $F(t) \le (F''(0) + \varepsilon)t^2/2$, for all $|t| \le M$. For all $0 < \lambda < M$, let now $u_\lambda(x) := \lambda u(x)$. A direct calculation using (1.6.16) shows that

$$\begin{aligned}
J_q(u_\lambda) &\le \lambda^2 \frac{\pi^2}{4T}\left(\frac{\pi^2}{T^2} - q\right) + \int_0^T F(u_\lambda)dx \\
&\le \lambda^2 \frac{\pi^2}{4T}\left(\frac{\pi^2}{T^2} - q\right) + \frac{F''(0) + \varepsilon}{2}\lambda^2 \int_0^T u(x)^2 dx \\
&\le \frac{\lambda^2}{4}\left[\frac{\pi^2}{T}\left(\frac{\pi^2}{T^2} - q\right) + (F''(0) + \varepsilon)T\right] =: \lambda^2 A.
\end{aligned}$$

Now, $A < 0$ amounts to $z^2 - qz + (F''(0) + \varepsilon) < 0$ having a positive solution $z = \pi^2/T^2$, which holds as long as $q^2 - 4(F''(0) + \varepsilon) > 0$. Letting $\varepsilon \to 0$ completes the proof. □

Remark 1.6.3. *Clearly, the previous lemma also provides with a precise interval of possible periods* T *for which* $\inf_{H_T} J_q < 0$. *Indeed, the thesis holds for all* T *such that*

$$\frac{q - \sqrt{q^2 - 4F''(0)}}{2} < \frac{\pi^2}{T^2} < \frac{q + \sqrt{q^2 - 4F''(0)}}{2}.$$

Thus, in the degenerate case $F''(0) = 0$, *we see that, for all sufficiently large* T *(precisely, for* $qT^2 > \pi^2$*), the thesis of the previous lemma holds. This will be essential in the study of the asymptotic behavior, as* $q \downarrow 0$, *of solutions to the S-H equation.*

Lemma 1.6.6. *Suppose that* $F \in C^2$ *satisfies* $F \ge 0$ *and*

$$\liminf_{|t|\to+\infty} \frac{F(t)}{t^2} \ge \alpha, \quad \textit{for some } \alpha \in]0, +\infty[.$$

Then, J_q *is bounded from below on* H_T, *for any* $q < \sqrt{2\alpha}$.

Proof. Clearly, we can suppose that $q > 0$, otherwise $J_q \ge 0$ trivially. Since $J_q(u) \ge -\frac{q}{2}\|u'\|_2^2$, it suffices to bound from above $\|u'\|_2$, for any $u \in H_T$ such that $J_q(u) \le 0$. Therefore, we can suppose that $\|u'\|_2 \ne 0$ and

$$\int_0^T |u''|^2\,dx + 2\int_0^T F(u)\,dx \le q\int_0^T |u'|^2\,dx. \tag{1.6.17}$$

In particular, it holds $\|u''\|_2^2 \le q\|u'\|_2^2$ which, inserted into (1.6.14), gives

$$\int_0^T |u'|^2\,dx \le \left(\int_0^T u^2 dx\right)^{1/2}\left(q\int_0^T |u'|^2 dx\right)^{1/2},$$

i.e., being $\|u'\|_2 \neq 0$,

$$\int_0^T |u'|^2 dx \leq q \int_0^T u^2 dx.$$

For any $\theta \in\ ]0,1[$, let $M > 0$ be such that $F(t) \geq \theta\alpha t^2$, for all $|t| > M$. From the previous displayed inequality and (1.6.17), we obtain

$$\begin{aligned}
\int_0^T |u'|^2 dx &\leq q\Big(\int_{\{|u|>M\}} u^2 dx + \int_{\{|u|\leq M\}} u^2\, dx\Big) \\
&\leq q\Big(\int_{\{|u|>M\}} \frac{F(u)}{\theta\alpha} dx + M^2 |\{|u(x)| \leq M\}|\Big) \\
&\leq q\Big(\int_0^T \frac{F(u)}{\theta\alpha} dx + M^2 T\Big) \\
&\leq q\Big(\frac{q}{2\theta\alpha} \int_0^T |u'|^2 dx + M^2 T\Big),
\end{aligned}$$

which implies

$$\Big(1 - \frac{q^2}{2\theta\alpha}\Big) \int_0^T |u'|^2 dx \leq q M^2 T.$$

If $q^2 \leq 2\theta\alpha$, then $1 - \frac{q^2}{2\theta\alpha} > 0$, and $\|u'\|_2$ is universally bounded. Being $\theta \in\ ]0,1[$ arbitrary, we obtain the claim. $\square$

Theorem 1.6.5. *Let F satisfy* (1.6.2)*. For any $q > 0$ such that*

$$4F''(0) < q^2 < 2 \liminf_{|t|\to+\infty} \frac{F(t)}{t^2},$$

there exists a nontrivial periodic solution to (1.6.1)*.*

Proof. By the previous two lemmas, $-\infty < \inf_{H_T} J_q < 0$, for some $T > 0$. Moreover, the proof of Lemma 1.6.6 shows that there exists a constant $C = C(q, T, F)$ such that

$$J_q(u) \leq 0 \quad \Rightarrow \quad \|u'\|_2^2 \leq C,$$

which implies, for any $u \in \{u \in H_T : J_q(u) \leq 0\}$, that

$$\|u''\|_2^2 \leq 2J_q(u) + q\|u'\|_2^2 \leq qC.$$

Finally, we can assume that

$$\liminf_{|t|\to+\infty} \frac{F(t)}{t^2} > 0$$

(otherwise there is nothing to prove), and thus there exists $M > 0$ such that $F(t) \geq \alpha t^2/2$, for $|t| \geq M$. Still for u such that $J_q(u) \leq 0$, it holds

$$\begin{aligned}
\|u\|_2^2 &\leq \int_{\{|u|\leq M\}} u^2\, dx + \frac{2}{\alpha} \int_{\{|u|\geq M\}} F(u)\, dx \\
&\leq M^2 T + \frac{2}{\alpha} \int_0^T F(u)\, dx \\
&\leq M^2 T + \frac{2}{\alpha}\Big(J_q(u) + q\|u'\|_2^2\Big) \\
&\leq M^2 T + \frac{2qC}{\alpha}.
\end{aligned}$$

Hence, $\{u \in H_T : J_q(u) \leq 0\}$ is bounded, and thus weakly sequentially relatively compact. Now, Lemma 1.6.1 provides the existence of a minimum $\bar{u}$, which is nontrivial, due to $J_q(\bar{u}) < 0$. Finally, Proposition 1.6.1 implies that the $2T$-periodic even extension of $\bar{u}$ is a solution to (1.6.1). $\square$

1.6.3 Asymptotic behavior

In this section we discuss the asymptotic behavior, as $q \downarrow 0$, of the periodic solutions of (1.6.1) obtained in Theorem 1.6.5. Clearly, in order to allow $q \downarrow 0$, we will assume in the following that $F''(0) = 0$. Upon vertical and horizontal translations of the potential F, we can suppose that $0 = F(0) = \min_{\mathbb{R}} F$. We define

$$\tilde{F}(t) := \min\{F(t), F(-t)\}, \qquad H(t) = \left(\tilde{F}(\sqrt{|t|})\right)^*. \tag{1.6.18}$$

Lemma 1.6.7. *Suppose that $F \in C^2$ satisfies*

$$\operatorname{Argmin}(F) = \{0\}, \tag{1.6.19}$$

$$\liminf_{|t| \to +\infty} \frac{F(t)}{t^2} > 0, \tag{1.6.20}$$

$$F''(0) = 0. \tag{1.6.21}$$

Then H, defined in (1.6.18)*, is an even convex function such that*

$$\varphi(t) := \begin{cases} \frac{H(t)}{t} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0 \end{cases}$$

is continuous and strictly increasing.

Proof. Since $G(t) := \widetilde{F}(\sqrt{|t|})$ is even, it follows immediately that H is even. From (1.6.20) we can find $\lambda > 0$ such that $F(x) \geq \lambda |x|^2$, for any sufficiently large $|x|$, which implies that $\widetilde{F}(\sqrt{t}) \geq \lambda |t|$, for sufficiently large $|t|$. Therefore, G satisfies (1.6.3) and thus Lemma 1.6.2 provides $\operatorname{Argmin}(H) = \operatorname{Argmin}(G) = \operatorname{Argmin}(F) = \{0\}$, by assumption. Moreover, by construction

$$H(t) \leq \widetilde{F}(\sqrt{|t|}) \leq F(\sqrt{|t|}),$$

so that (1.6.21) implies

$$\lim_{t \to 0} \frac{H(t)}{t} = 0. \tag{1.6.22}$$

In particular, φ is continuous. We then observe that, for $t \neq 0$,

$$\varphi(t) = \frac{H(t) - H(0)}{t - 0},$$

and the convexity of H implies that φ is non-decreasing. To prove strict monotonicity, suppose, by contradiction, that $\varphi(t_1) = \varphi(t_2)$ for some $t_1 < t_2$. Since $\varphi(t)t > 0$ for $t \neq 0$, we can assume, without loss of generality, that $t_1 > 0$. Since φ is non-decreasing, we infer $\varphi(t) = \varphi(t_1) =: \lambda > 0$, for all $t \in [t_1, t_2]$, i.e., $H(t) = \lambda t$, for $t \in [t_1, t_2]$. Therefore, λt is a support line for H and, by convexity, we obtain $H(t) \geq \lambda t$, for all $t > 0$. This in turn implies that $H(t) \geq \lambda |t|$, being H even, and we reach a contradiction through (1.6.22). ☐

Theorem 1.6.6. *Suppose that* (1.6.19)*,* (1.6.20)*,* (1.6.21) *hold and $qT^2 > \pi^2$. Then, the minimum $u_{q,T}$ of J_q on H_T exists, is nontrivial and satisfies*

$$\operatorname{Osc}(u_{q,T})^2 \leq qT^2 \varphi^{-1}\left(\frac{q^2}{2}\right), \tag{1.6.23}$$

where φ is the function given in Lemma 1.6.7.

Proof. Clearly, (1.6.19) implies, eventually adding a constant, that (1.6.2) holds. Thus, Theorem 1.6.5 and Remark 1.6.3 show the existence part of the statement. We let, for simplicity, $u := u_{q,T}$. From $\inf_{H_T} J_q \le 0$ we deduce

$$\int_0^T |u''|^2 \, dx + 2 \int_0^T F(u) \, dx \le q \int_0^T |u'|^2 \, dx. \tag{1.6.24}$$

Let H be given by (1.6.18). By the previous lemma, we infer in particular that $\lim_{t \to +\infty} H(t) = +\infty$, so that H is invertible on $[0, +\infty[$. By Jensen's inequality we have

$$H\Big(\fint_0^T u^2 dx\Big) \le \fint_0^T H(u^2) dx \le \fint_0^T F(u) dx,$$

hence

$$\int_0^T u^2 dx \le T H^{-1}\Big(\fint_0^T F(u) dx\Big).$$

Therefore, from (1.6.14) and (1.6.24) we have

$$\int_0^T |u'|^2 dx \le T^{1/2} \Big[H^{-1}\Big(\fint_0^T F(u) dx\Big)\Big]^{1/2} \Big(q \int_0^T |u'|^2 dx\Big)^{1/2},$$

and simplifying we get

$$\int_0^T |u'|^2 dx \le q T H^{-1}\Big(\fint_0^T F(u) dx\Big).$$

Since H^{-1} is increasing on $[0, +\infty[$, using again (1.6.24) we have

$$\frac{1}{qT} \int_0^T |u'|^2 dx \le H^{-1}\Big(\frac{q}{2} \fint_0^T |u'|^2 dx\Big).$$

Letting $z = z(q, T) := \frac{1}{qT} \int_0^T |u'|^2 dx$, and $\varphi(z) = H(z)/z$, the last inequality reads

$$\varphi(z) \le \frac{q^2}{2}.$$

By Lemma 1.6.7, φ is invertible on $\{|\varphi| \le q^2/2\}$, for sufficiently small q, so that

$$\int_0^T |u'|^2 dx \le qT \varphi^{-1}\Big(\frac{q^2}{2}\Big) \tag{1.6.25}$$

and, using the standard inequality

$$\mathrm{Osc}(u)^2 \le T \int_0^T |u'|^2 \, dx,$$

we obtain (1.6.23). □

Corollary 1.6.1. *Suppose F satisfies* (1.6.19), (1.6.20) *and* (1.6.21). *Then, for any $q > 0$, there exists $T(q) > 0$ such that the minimum $u_{q,T(q)}$ of J_q on H_T exists, is nontrivial and satisfies*

$$\lim_{q \to 0} \|u_{q,T(q)}\|_\infty = 0.$$

Proof. For any $q > 0$ we choose $T(q)$ such that

$$T(q) > \frac{\pi}{\sqrt{q}}, \qquad \lim_{q\to 0^+} qT^2(q)\varphi^{-1}\Big(\frac{q^2}{2}\Big) = 0$$

(e.g., $\pi^2 < qT^2(q) \le K$, for some $K > \pi^2$, suffices). The previous theorem provides the nontrivial solution $u_{q,T(q)}$ satisfying $\mathrm{Osc}(u_{q,T(q)}) \to 0$ and, to complete the proof, it suffices to show that $u_{q,T(q)}(0) \to 0$. Suppose by contradiction that $u_{q_n,T(q_n)}(0) \ge \varepsilon > 0$, for some $q_n \to 0^+$, and let $u_n = u_{q_n,T(q_n)}$, $T_n = T(q_n)$. By (1.6.24) and (1.6.25) it holds

$$\int_0^{T_n} F(u_n)\,dx \le \frac{q_n}{2}\int_0^{T_n} |u_n'|^2\,dx \le \frac{1}{2} q_n^2 T_n^2 \varphi^{-1}\Big(\frac{q_n^2}{2}\Big) \to 0.$$

Since $\mathrm{Osc}(u_n) \to 0$, for sufficiently large n it holds $u_n(x) \ge u_n(0) - \mathrm{Osc}(u_n) > \varepsilon/2$ for any x, which implies

$$\int_0^{T_n} F(u_n)\,dx \ge \frac{\varepsilon}{2} T_n > \frac{\pi\varepsilon}{2\sqrt{q_n}} \to +\infty,$$

which contradicts the previous displayed estimate. □

The case of homogeneous potentials

Suppose now that F is homogeneous, e.g., $F(u) = \frac{|u|^r}{r}$. We thus focus on the model equation

$$u'''' + qu'' + |u|^{r-1}u = 0. \tag{1.6.26}$$

Theorem 1.6.7. *Let $r > 2$. If (u_n) is a sequence of bounded solutions to* (1.6.26)*, for some $q_n \downarrow 0$, then*

$$\lim_n \|u_n\|_\infty = 0.$$

Proof. Suppose u is any bounded nontrivial solution to (1.6.26) and let $v_\lambda(x) := u(\lambda x)$, for all $\lambda > 0$. Observe that $\mathrm{Osc}(v_\lambda) = \mathrm{Osc}(u)$ and that v_λ solves

$$v_\lambda'''' + q\lambda^2 v_\lambda'' + \lambda^4 |v_\lambda|^{r-2} v_\lambda = 0.$$

Furthermore, if

$$w_\lambda(x) := \frac{v_\lambda(x)}{\mathrm{Osc}(v_\lambda)} = \frac{v_\lambda(x)}{\mathrm{Osc}(u)},$$

w_λ solves

$$w_\lambda'''' + q\lambda^2 w_\lambda'' + \lambda^4 \mathrm{Osc}(u)^{r-2} |w_\lambda|^{r-2} w_\lambda = 0$$

for all $\lambda > 0$. Choosing $\lambda^4 = \mathrm{Osc}(u)^{2-r}$, we obtain

$$w'''' + \frac{q}{\mathrm{Osc}(u)^\gamma} w'' + |w|^{r-2} w = 0.$$

Applying this scaling argument to $u = u_n$, $q = q_n$ and letting $\mathrm{O}_n = \mathrm{Osc}(u_n)$, $w_n = u_n/\mathrm{O}_n$, $\gamma = \frac{r}{2} - 1 > 0$, we get

$$w_n'''' + \frac{q_n}{\mathrm{O}_n^\gamma} w_n'' + |w_n|^{r-2} w_n = 0. \tag{1.6.27}$$

We claim that

$$\liminf_n \frac{q_n}{\mathrm{O}_n^\gamma} \ge \beta > 0. \tag{1.6.28}$$

Arguing by contradiction, suppose that, up to a not relabeled subsequence,

$$\lim_n \frac{q_n}{\mathrm{O}_n^\gamma} = 0. \tag{1.6.29}$$

In particular, we can suppose, without loss of generality, that $q_n/\mathrm{O}_n^\gamma \leq 1$. From (1.6.27) we have

$$|w_n''''| \leq |w_n''| + |w_n|^{r-1}$$

which, using the L^∞-interpolation inequality $\|u''\|_\infty \leq C\sqrt{\|u\|_\infty \|u''''\|_\infty}$ and Young's inequality, implies

$$\|w_n\|_{C^4(\mathbb{R})} \leq C\|w_n\|_\infty. \tag{1.6.30}$$

We now want to prove that $\|w_n\|_\infty \leq 1$. On the contrary, suppose that $\|w_n\|_\infty > 1$. Then, without loss of generality, we can suppose that $\sup w_n > 1$ and, since $\mathrm{Osc}(w_n) = 1$, we have $w_n(x) > 0$ for all $x \in \mathbb{R}$. This implies, through (1.6.27), that

$$\left(w_n'' + \frac{q_n}{\mathrm{O}_n^\gamma} w_n\right)'' = -|w_n|^{r-2} w_n < 0 \quad \text{anywhere in } \mathbb{R}, \tag{1.6.31}$$

hence the function $w_n'' + \frac{q_n}{\mathrm{O}_n^\gamma} w_n$ is concave. Since it is also bounded, it must be constant, contradicting the strict inequality in (1.6.31).

Let I_n be the interval $w_n(\mathbb{R})$. From

$$|I_n| = 1, \qquad I_n \subseteq [-1, 1],$$

a standard compactness argument shows that (up to a not relabeled subsequence) there exists an interval J of length $1/2$ such that $J \subseteq \mathrm{int}(I_n)$, for all n. Let $\lambda \in J \setminus \{0\}$. Being $\lambda \in w_n(\mathbb{R})$, let x_n be such that $w_n(x_n) = \lambda$, and let $v_n(x) := w_n(x + x_n)$. Clearly, v_n solves (1.6.27) and satisfies $\mathrm{Osc}(v_n) \equiv 1$. Moreover, (1.6.30) and $\|v_n\|_\infty = \|w_n\|_\infty \leq 1$ show, by Ascoli-Arzelà's Theorem, that (v_n) is a compact sequence in $C^3_{\mathrm{loc}}(\mathbb{R})$, which we can suppose converges to some $v_0 \in C^3_{\mathrm{loc}}(\mathbb{R})$. Passing to the limit in the weak formulation of (1.6.27) and using (1.6.29) we obtain

$$v_0'''' + |v_0|^{r-2} v_0 = 0$$

weakly, and thus strongly. Now, Theorem 1.6.2 implies $v_0 \equiv 0$, contradicting

$$v_0(0) = \lim_n v_n(0) = \lim_n w_n(x_n) = \lambda \neq 0.$$

Thus (1.6.28) is proved, implying that, for any sufficiently large n, it holds

$$\mathrm{Osc}(u_n)^\gamma \leq \frac{2}{\beta} q_n,$$

which proves that

$$\mathrm{Osc}(u_n) \to 0. \tag{1.6.32}$$

It remains to prove that $u_n(0) \to 0$. The argument is the same as before and we only sketch it. If $u_n(0) \geq \varepsilon > 0$, then, by (1.6.32), for sufficiently large n it holds $u_n(x) \geq \frac{\varepsilon}{2} > 0$, for all $x \in \mathbb{R}$, which implies

$$(u_n'' + q_n u_n)'' = -|u|^{r-2} u < 0 \quad \text{everywhere.}$$

Being $u_n'' + q_n u_n$ bounded, it must be constant, contradicting the previous strict inequality. $\square$

1.7 Further developments

1. We studied equation (1.3.1) in the case $q \in \mathbb{R}$. It would be interesting to know if such results can be still obtained if we consider a function $g = g(u)$ instead of q, with suitable properties, thus leading to the equation

$$u'''' - g(u)u'' - \frac{1}{2}g'(u)u'^2 + F'(u) = 0. \tag{1.7.1}$$

   This was already made in [19, 20, 61] for the existence of heteroclinic solutions of (1.7.1) in the case when $g \in C^1(\mathbb{R})$ is a nonnegative function.

2. It is worth noting that, while (1.5.4) and (1.6.9) are roughly complementary conditions to establish (or rule out) existence of global nontrivial solutions to (1.3.1), a truly necessary and sufficient condition is still missing. It is worth comparing with the well known Keller-Osserman necessary and sufficient condition

$$\int_*^{\pm\infty} |F(t)|^{-\frac{1}{2}}\, dt < +\infty$$

   for the blow-up of solutions to the second order ODE $u'' - F'(u) = 0$. Even for $q = 0$, no such integral optimal condition is known for (1.3.1).

3. Equation (1.3.1) has a natural extension in $\mathbb{R}^N$ to the following biharmonic equation

$$\Delta^2 u + q\Delta u + F'(u) = 0, \qquad q \in \mathbb{R}, \tag{1.7.2}$$

   where Δ^2 is the bi-Laplace operator. It would be interesting to see if our hypotheses on the functional F allow us to obtain existence of bounded solutions to equation (1.7.2).

Chapter 2

Elliptic problems involving the critical exponent

2.1 Introduction

One of the most useful tools in the theory of partial differential equation and in particular in the calculus of variations is represented by the Sobolev embedding theorems, also called Sobolev inequalities, which have received great attention from a large number of authors. Let p be a number such that $1 \leq p < \infty$. We denote by $L^p(\mathbb{R}^N)$, $L^p(\mathbb{R}^N;\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$ the usual Lebesgue and Sobolev spaces equipped with the norms $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$ given by

$$\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{1}{p}}, \qquad \|\nabla u\|_p = \left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{\frac{1}{p}},$$

$$\|u\|_{1,p} = \left(\int_{\mathbb{R}^N} |\nabla u|^p dx + \int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{1}{p}}.$$

Similar definitions hold if $\mathbb{R}^N$ is replaced by an arbitrary domain Ω. In particular, on the boundary $\partial\Omega$, we use the $(N-1)$-dimensional Hausdorff (surface) measure denoted by σ. Then, in a natural way we can define the Lebesgue spaces $L^s(\partial\Omega)$ with $1 \leq s \leq \infty$ and the norms $\|\cdot\|_{s,\partial\Omega}$ which are given by

$$\|u\|_{s,\partial\Omega} = \left(\int_{\partial\Omega} |u|^s d\sigma\right)^{\frac{1}{s}} \quad (1 \leq s < \infty), \qquad \|u\|_{\infty,\partial\Omega} = \operatorname*{ess\,sup}_{\partial\Omega} |u|.$$

Probably the simplest starting point for a Sobolev inequality is to find an exponent $q > 0$ such that

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^N)} \tag{2.1.1}$$

for all u in an appropriate subfamily of $W^{1,p}(\mathbb{R}^N)$. Assume for simplicity that $u \in C_c^1(\mathbb{R}^N)$ and for $r > 0$ define the rescaled function

$$u_r(x) := u(rx), \quad x \in \mathbb{R}^N.$$

If (2.1.1) holds for u_r, we get

$$\begin{aligned}
\left(\int_{\mathbb{R}^N} |u(rx)|^q dx\right)^{1/q} &= \left(\int_{\mathbb{R}^N} |u_r(x)|^q dx\right)^{1/q} \\
&\leq C\left(\int_{\mathbb{R}^N} |\nabla u_r(x)|^p dx\right)^{1/p} \\
&= C\left(r^p \int_{\mathbb{R}^N} |\nabla u(rx)|^p dx\right)^{1/p},
\end{aligned}$$

or, equivalently, after the change of variables $y := rx$,

$$\left(\frac{1}{r^N}\int_{\mathbb{R}^N}|u(y)|^q dy\right)^{1/q} \leq C\left(\frac{r^p}{r^N}\int_{\mathbb{R}^N}|\nabla u(y)|^p dy\right)^{1/p},$$

that is

$$\left(\int_{\mathbb{R}^N}|u(y)|^q dy\right)^{1/q} \leq Cr^{1-\frac{N}{p}+\frac{N}{q}}\left(\int_{\mathbb{R}^N}|\nabla u(y)|^p dy\right)^{1/p}.$$

If $1-\frac{N}{p}+\frac{N}{q}>0$, let $r\to 0^+$ to conclude that $u\equiv 0$, while if $1-\frac{N}{p}+\frac{N}{q}<0$, let $r\to+\infty$ to conclude again that $u\equiv 0$. Hence, the only possible case is when

$$\frac{N}{q}=\frac{N}{p}-1.$$

So in order for q to be positive, we need $p<N$, in which case

$$q=p^* := \frac{Np}{N-p}.$$

The number p^* is called the *Sobolev critical exponent.* Inequality (2.1.1) is justified by the following seminal theorem.

Theorem 2.1.1 (Sobolev-Gagliardo-Nirenberg's embedding theorem)**.** *Let $1\leq p<N$. Then, there exists a constant $C=C(N,p)>0$ such that for every function $u\in L^p(\mathbb{R}^N)$ vanishing at infinity it holds*

$$\left(\int_{\mathbb{R}^N}|u(x)|^{p^*}dx\right)^{1/p^*} \leq C\left(\int_{\mathbb{R}^N}|\nabla u(x)|^p dx\right)^{1/p}.$$

In particular, $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for all $p\leq q\leq p^*$.

A quite subtle issue concerning inequality (2.1.1) is that of the optimal constant C. When $p=1$, this question was settled independently by Federer and Fleming [48] and Maz'ya [94] at the beginning of 1960. The problem for $p\in(1,N)$ was solved only about fifteen years later, again independently in two papers by Aubin [10] and Talenti [131]. Talenti, in particular, evaluated the following expression

$$C=\sup\frac{\|u\|_q}{\|\nabla u\|_p}, \tag{2.1.2}$$

where the supremum is taken in the class of all not identically zero smooth functions u which decay rapidly at infinity. Incidentally, the supremum in question does not change if the involved functions are restricted to have their support in some fixed open set. If $1<p<N$ and $q=p^*$, then (2.1.1) can be proved for functions u in $C_0^1(\mathbb{R}^N)$ by using the straightforward representation formula

$$u(x)=-\frac{\Gamma(N/2)}{2\pi^{N/2}}\int_{\mathbb{R}^N}|x-y|^{1-N}\sum_{k=1}^{N}\frac{x_k-y_k}{|x-y|}\frac{\partial u}{\partial x_k}(y)dy$$

and by applying to the right-hand side an N-dimensional version of a theorem of Hardy-Littlewood concerning fractional integrals. This is the method of Sobolev [124, 125], but unfortunately it neither gives the exact value of the best constant C nor explicit estimates for C. On the contrary, Talenti in [131] gave the best value of C, proving that

$$C=\pi^{-1/2}N^{-1/p}\left(\frac{p-1}{N-p}\right)^{1-1/p}\left\{\frac{\Gamma(1+N/2)\Gamma(N)}{\Gamma(N/p)\Gamma(1+N-N/p)}\right\}^{1/N}.$$

Moreover, the ratio in (2.1.2) attains its maximum value C on functions u of the form

$$u(x) = [a + b|x|^{p/(p-1)}]^{1-N/p},$$

where $|x| = (x_1^2 + \cdots + x_N^2)^{1/2}$ and a, b are positive constants. One can also ask on the validity of Theorem 2.1.1 if one replaces $\mathbb{R}^N$ with a domain Ω. It is easily seen (cf. [82, Exercise 11.7]) that Theorem 2.1.1 fails for an arbitrary Ω, since its validity is intimately related to the regularity of the boundary. But, if Ω is a bounded domain with a Lipschitz $\partial\Omega$, then Theorem 2.1.1 is still true, provided that $u|_{\partial\Omega} = 0$ (cf. [82, Corollary 11.9]).

Another important question related to the Sobolev inequalities on bounded domains is the compactness of the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, as the next result states.

Theorem 2.1.2 (Rellich-Kondrachov's theorem)**.** *Let $1 \leq p < N$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Let $(u_n) \subseteq W^{1,p}(\Omega)$ be a bounded sequence. Then, there exist a subsequence (u_{n_k}) of (u_n) and a function $u \in L^{p^*}(\Omega)$ such that $u_{n_k} \to u$ in $L^q(\Omega)$ for all $1 \leq q < p^*$.*

The most important consequence of this theorem is that the embedding

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \tag{2.1.3}$$

is actually *non compact*. The lack of compactness for the embedding (2.1.3) has important consequences in the theory of partial differential equations. Indeed, the main difficulty is that the functional associated to a prescribed problem *does not satisfy the Palais-Smale condition*. Hence there are serious difficulties when trying to find its critical points by standard variational methods. A brief description of this phenomenon is provided below.

2.2 Boundary value problems with critical exponent

The aim of this section is to present some boundary value problems which involve the critical exponents both in the domain and on the boundary. This would not be an exhaustive list of such problems, instead it is only an attempt to give to the reader an idea of which are the difficulties when dealing with problems where the critical exponent is involved. In what follows, by S we mean the best constant for the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, for every $1 < p < N$.

2.2.1 Dirichlet condition

Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 3$. We are concerned with the following nonlinear elliptic problem

$$\begin{aligned} -\Delta u &= u^r + \lambda u && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.2.1}$$

where $r = 2^* - 1$. Solutions of (2.2.1) correspond to critical points of the functional

$$\Phi_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega |u|^2 dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx.$$

Owing to Theorem 2.1.2, we already know that the functional Φ_λ does not satisfy the (PS) condition (that is, if $(u_j) \subseteq H^1(\Omega)$ is a sequence such that $\Phi_\lambda(u_j) \to c \in \mathcal{I}$ and $\Phi'_\lambda(u_j) \to 0$ in $H^{-1}(\Omega)$ as $j \to +\infty$, then there exists a subsequence of (u_j) which converges weakly to $u_0 \neq 0$, and u_0 is a critical point of $\Phi_\lambda(u)$). Therefore, the problem of finding critical points of Φ_λ by standard variational methods arises. In fact, there is a sharp contrast between the case $r < 2^* - 1$ and the case $r = 2^* - 1$. Many existence results for problem (2.2.1) are known when $r < 2^* - 1$ (see the review article by P.L. Lions [85] and its references). On the other hand, a well-known nonexistence result of Pohozaev [110] asserts that if Ω is starshaped there is *no* solution of problem (2.2.1) when $\lambda \equiv 0$.

In [23] the authors used a different viewpoint, namely looking for critical points of the functional $u \mapsto \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega u^2 dx$ on the sphere $\|u\|_{2^*} = 1$, which are seen to satisfy the equation

$$-\Delta u - \lambda u = \mu u^{2^*-1},$$

where μ is a Lagrange multiplier. After 'stretching' the Lagrange multiplier they obtained a solution of (2.2.1) by proving that for suitable λ's

$$\inf_{\substack{u \in H_0^1 \\ \|u\|_{2^*}=1}} \left\{ \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega u^2 dx \right\} \text{ is achieved.} \tag{2.2.2}$$

The major difficulty in proving (2.2.2) stems from the fact that the function $u \mapsto \|u\|_{2^*}$ is *not* continuous under weak convergence in $H_0^1(\Omega)$. The decisive device in order to overcome this lack of compactness is to establish that for suitable λ's we have

$$\inf_{\substack{u \in H_0^1 \\ \|u\|_{2^*}=1}} \left\{ \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega u^2 dx \right\} < \inf_{\substack{u \in H_0^1 \\ \|u\|_{2^*}=1}} \left\{ \int_\Omega |\nabla u|^2 dx \right\} \equiv S.$$

Their arguments are mainly inspired by Aubin [9], and have led to the conclusion that, if λ_1 is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$, there exists a constant $\lambda^* \in [0, \lambda_1)$ such that (2.2.1) has a positive solution for $\lambda \in (\lambda^*, \lambda_1)$. Moreover, if $N \geq 4$, then $\lambda^* = 0$.

When $p \neq 2$, problem (2.2.1) was studied in [60]. Suppose that λ_1 is the best Poincaré constant in $W_0^{1,p}(\Omega)$, that is

$$\lambda_1 = \max\left\{ \rho > 0 : \int_\Omega |\nabla u|^p dx \geq \rho \int_\Omega |u|^p dx, \quad \forall\, u \in W_0^{1,p}(\Omega) \right\}.$$

They proved that there exists a solution $u \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$, if $1 < p^2 \leq N$ and $0 < \lambda < \lambda_1$. The ideas for proving this result are essentially the same as those of [23, 9, 137]: they first define S_λ as in (2.2.2), and proved that $S_\lambda < S$ for every $\lambda > 0$. If $0 < S_\lambda < S$, then the infimum is achieved by some u which gives (after some stretching) a solution of the equation. The most delicate point is to obtain *a-priori* $C^{1,\alpha}(\overline{\Omega})$ estimates on approximating solutions and for which they used Tolksdorf's method [134].

Let now $\Omega \subseteq \mathbb{R}^N$ be a smooth bounded domain, let $1 < q < p < N, \lambda > 0$ and consider the following problem

$$\begin{aligned} \Delta_p u &= |u|^{p^*-2}u + \lambda |u|^{q-2}u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.2.3}$$

Evidently, solutions of (2.2.3) are critical points of the functional

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{p^*}\int_\Omega |u|^{p^*} dx - \frac{\lambda}{q}\int_\Omega |u|^q dx.$$

In [51] the authors proved the existence of at least two solutions for problem (2.2.3) by essentially using three main tools: the local (PS) condition, the Mountain Pass lemma and energy estimates.

2.2.2 Homogeneous Robin condition

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with C^1 boundary and consider the following problem

$$\begin{aligned} -\Delta u &= u^{2^*-1} + f(x,u) && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &+ \alpha(x) u = 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.2.4}$$

Here n is the unit outward normal to $\partial\Omega$, α is a nonnegative function and $f(x,u)$ is a lower order perturbation of u^{2^*-1} at infinity such that $f(x,0)=0$. It was proven by Wang [144] that the weak solutions of (2.2.4) are equivalent to the nonzero critical points of the functional

$$J(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{2^*}(u^+)^{2^*} - F(x,u)\right) dx + \frac{1}{2}\int_{\partial\Omega} \alpha(x) u^2 d\sigma,$$

where $F(x,u) = \int_0^u f(x,t)dt$. Although the standard variational methods are not working in this case, Wang applied a general existence theorem based on a variant of the Mountain Pass lemma. Indeed, he proved that $J(u)$ satisfies the $(PS)_c$ condition in a weak sense for $c \in (0, (1/(2N)S^{N/2}) =: \mathcal{I}$.

Consider now the following boundary value problem

$$\begin{aligned} -\Delta u &= |u|^{2^*-2}u && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &+ u + |u|^{2_*-2}u = 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.2.5}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 4$, is a bounded domain with a smooth C^2 boundary and $2_* := 2(N-1)/(N-2)$ is the limiting exponent for the embedding of $H^1(\Omega)$ into $L^q(\partial\Omega)$. The main interest in studying this kind of problem rests in the presence of critical exponents both in the equation and in the nonlinear boundary condition. Reasoning as in the previous cases, weak solutions of (2.2.5) are the critical points of the following C^1 functional

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx + \frac{1}{2}\int_{\partial\Omega} |u|^2 d\sigma + \frac{1}{2_*}\int_{\partial\Omega} |u|^{2_*} d\sigma.$$

In [109], Pierotti and Terracini made careful analysis of the features of a (PS) sequence for F, in order to overcome its lack of compactness. Following the same point of view adopted by Struwe in [128] for the Dirichlet problem, they determined 'safe' sublevels where standard critical point theorems apply obtaining, as a consequence, the existence of critical points for F.

2.3 Physical and geometrical background

Our motivation for investigating the aforementioned problems comes from the fact that they resemble some variational problems in geometry and physics where lack of compactness also occurs. The most notorious example, for historical motivations, is the *Yamabe's problem*.

2.3.1 Yamabe's problem

Riemannian differential geometry originated in attempts to generalize the successful theory of compact surfaces. From the earliest days, *conformal* changes of metric (i.e., multiplication of the metric by a positive function) have played an important role in surface theory. A well-known open question is to determine whether a given compact Riemannian manifold is conformally equivalent to one of constant scalar curvature. This problem is known as the *Yamabe's problem* because it was stated in 1960 by Yamabe [151], and can be formulated as follows. Given a smooth, compact manifold M of dimension $N \geq 3$ with a Riemannian metric g, does there exist a metric g' conformal to g for which the scalar curvature of g' is constant?

While Yamabe's paper claimed to solve the problem in the affirmative, Trudinger [137] found that this paper was seriously incorrect, and improved it in the case of nonpositive scalar curvature. Progress was made in the case of positive scalar curvature by Aubin [11], who solved the problem for a manifold M such that $\dim M \geq 6$. Up until this time, Aubin's method has given no information on the Yamabe's problem in dimensions $3, 4$ and 5. Moreover, his method exploited only the local geometry of M in a small neighborhood of a point, and hence could not be used on a conformally flat manifold where the Yamabe's problem is clearly global.

In [118] Schoen introduced a new global idea for this problem. More specifically, he asserted the existence of a positive solution u of M of the equation

$$\Delta u - \frac{N-2}{4(N-1)} R u + u^{2^*-1} = 0, \tag{2.3.1}$$

where $R > 0$ is the scalar curvature of M. By its intrinsic geometric meaning equation (2.3.1) is conformally invariant (see [12, Proposition, p. 126] or [129, page 194]). For this reason, the Yamabe's problem can be seen as a noncompact variational problem, for which the loss of compactness caused by the invariant action of its conformal group leads to possible spikes formation. To overcome this difficulty, the a-priori knowledge of the energy range where the Palais-Smale condition holds is helpful, and sometimes suffices to construct critical points. From this point of view, the problem was solved in complementing cases by Aubin [9] and Schoen [118].

2.3.2 Existence of extremal functions in functional inequalities

One of the main difficulties in the study of variational problems set in unbounded domains (the so-called limit-cases problems) is the possible loss of compactness caused by the invariance of $\mathbb{R}^N$ by the *non-compact group of dilations* and the *non-compact group of translations*.

The dilations invariance of $\mathbb{R}^N$ is a typical difficulty in the study of the existence of *extremal functions* in *functional inequalities*. Indeed, if A is a linear bounded operator between two Banach spaces E and F, one may consider the smallest positive constant C_0 such that the following inequality holds

$$\|Au\|_F \le C_0\|u\|_E \quad \forall\, u \in E. \tag{2.3.2}$$

If E, F are functional spaces, it is often the case that (2.3.2) is preserved if we perform a scale change, namely replacing $u(\cdot)$ by $u(\cdot/\sigma)$ for $\sigma > 0$. Moreover, one may ask whether the *best constant* C_0 is obtained for some u. This is equivalent to solving one of the following minimization problems

$$\begin{aligned} &\text{(a)}\ \min\{\|u\|_E : u \in E,\ \|Au\|_F = 1\},\\ \text{or}\quad &\text{(b)}\ \min\{-\|Au\|_F : u \in E,\ \|u\|_E = 1\}, \end{aligned} \tag{2.3.3}$$

and the invariance of (2.3.2) by scale changes is often reflected by the invariance of $\|\cdot\|_E$ or $\|\cdot\|_F$ by the changes $u(\cdot) \mapsto \sigma^{-\alpha}u(\cdot/\sigma)$, where $\alpha = \alpha(A, E, F)$. This invariance implies compactness defects on minimizing sequences of problems (2.3.3). Below we list some examples where such situations apply.

1. Let $1 \le p < N$. Then inequality (2.3.2) is the Sobolev embedding theorem where $E = W^{1,p}(\mathbb{R}^N)$, $F = L^{p^*}(\mathbb{R}^N)$ and A is the injection $E \hookrightarrow F$. The associated minimization problem is

$$\min\left\{\int_{\mathbb{R}^N} |\nabla u|^p dx : u \in E,\ \int_{\mathbb{R}^N} |u|^{p^*} dx = 1\right\}.$$

If we replace u by $\sigma^{-N/p^*}u(\cdot/\sigma)$ for any $\sigma > 0$, then the two functionals occurring in the above variational problem are preserved (note that this invariance is nothing else than the invariance of Sobolev inequalities with respect to scale changes).

2. Let now $0 < \mu < N, 1 < p < (N/(N-\mu))$ and let q satisfy $1/p+\mu/N = 1+1/q$. Then we may take (2.3.2) as the Hardy-Littlewood-Sobolev inequality

$$\|K * u\|_q \le C_0\|u\|_p, \quad \forall\, u \in L^p(\mathbb{R}^N),$$

where $K = 1/|x|^\mu$. The determination of the best constant C_0 is then equivalent to solve (2.3.3.b), where $E = L^p(\mathbb{R}^N), F = L^q(\mathbb{R}^N)$ and $Au = K * u$. Moreover, the two functionals are invariant by the transformation $u \mapsto \sigma^{-N/q}u(\cdot/\sigma)$, for any $\sigma > 0$.

3. Finally, let $1 \le p < N, N \ge 2$ and let q given by $q = p_* := \frac{p(N-1)}{N-p}$. Then (2.3.2) is the trace inequality when $E = \{u \in L^{p^*}(\mathbb{R}^{N-1} \times \mathbb{R}^+), \nabla u \in L^p(\mathbb{R}^{N-1} \times \mathbb{R}^+)\}$ equipped with the norm $\|\nabla u\|_p$, $F = L^{p_*}(\mathbb{R}^{N-1})$ and A is the linear operator such that, if $u \in \mathcal{D}(\mathbb{R}^N)$, then Au is the usual trace of u on $\mathbb{R}^{N-1} \times \{0\}$. For obvious reasons we still denote Au by u. Problem (2.3.3.a) then becomes

$$\min\left\{\int_{\mathbb{R}^{N-1}\times\mathbb{R}^+} |\nabla u|^p dx : u \in E,\ \int_{\mathbb{R}^{N-1}} |u(x',0)|^{p_*} dx' = 1\right\},$$

and both functionals are preserved if we replace u by $\sigma^{-N/p_*}u(\cdot/\sigma)$.

By *loss of compactness induced by the dilations group* we mean that, even if we know that there exists a minimum, the set of minima is not relatively compact in E. Indeed, if u is such a minimum then $u_\sigma := \sigma^{-\alpha}u(\cdot/\sigma)$ would still be a minimum for all $\sigma > 0$. Now if $\sigma \to +\infty$, u_σ converges weakly to 0 (which is *not* a minimum) and the probability $|u_\sigma|^q$ (or $|u_\sigma|^p$) either converges weakly as $\sigma \to 0$ to a Dirac mass or spreads out as $\sigma \to +\infty$. Moreover, the sets of minima are also translation invariant, and this still induce loss of compactness as well.

A general method to solve variational problems (with constraints) where such difficulties are encountered was presented in [86]. Roughly speaking, this method enables one to prove that *any minimizing sequence is relatively compact in* E up to a translation and a scale change. In particular there exists a minimum.

2.3.3 The Schrödinger equation

Consider the following equation

$$-\varepsilon^2\Delta u + a(x)u = u^{2^*-1} \qquad \text{in } \mathbb{R}^N \tag{2.3.4}$$

for $\varepsilon > 0$ small. Problem (2.3.4) arises in the search of standing waves for the nonlinear Schrödinger equation. Such a wave has the form $\psi(t,x) = \exp(-i\lambda\hbar^{-1}t)u(x)$ and represents the quantum mechanical probability amplitude for a given particle of unit mass to have position x at time t.

Schrödinger equations with critical growth terms arise in the context of magnetic fields [29, 7] and have also been established in fluid mechanics [74, 75], in the theory of Heidelberg ferromagnetism and magnus [76], in dissipative quantum mechanics [64] and in condensed matter theory [90]. Most of these papers are concerned with the concept of soliton solution. Following [64], we define one-soliton as a normalizable solution of nonlinear Schrödinger equation which vanishes at $x = \pm\infty$ and such that all its points are moving with the same constant velocity, thus preserving the shape of the wave in the course of time.

2.4 Regularity theory and existence theory

In this section we will focus on the solvability of certain classes of boundary value problems and related general properties of the corresponding solutions. The following fixed point result is the most often applied in the approach to the Dirichlet problem for quasilinear equations (see [58] for further informations).

Theorem 2.4.1. *Let T be a compact mapping of a Banach space B into itself, and suppose that there exists a constant M such that*

$$\|x\|_B < M \tag{2.4.1}$$

for all $x \in B$ and $\sigma \in [0,1]$ satisfying $x = \sigma Tx$. Then, T has a fixed point.

The previous theorem implies that if T is *any* compact mapping of a Banach space into itself (whether or not (2.4.1) holds), then for *some* $\sigma \in (0,1]$ the mapping σT has a fixed point. Furthermore, if the estimate (2.4.1) holds then σT has a fixed point for all $\sigma \in [0,1]$.

In order to apply Theorem 2.4.1 to the Dirichlet problem for quasilinear equations, we fix a number $\beta \in (0,1)$ and take the Banach space B to be the Hölder space $C^{1,\beta}(\overline{\Omega})$, where Ω is a bounded domain in $\mathbb{R}^N$. Let $\mathcal{Q}$ be the operator given by

$$\mathcal{Q} = a^{ij}(x,u,Du)D_{ij}u + b(x,u,Du)$$

and assume that $\mathcal{Q}$ is elliptic in $\overline{\Omega}$, that is, the coefficient matrix $[a^{ij}(x,z,\rho)]$ is positive for all $(x,z,\rho) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$. We also assume, for some $\alpha \in (0,1)$, that the coefficients $a^{ij}, b \in C^{\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N)$, that the boundary $\partial\Omega \in C^{2,\alpha}$ and that φ is a given function in $C^{2,\alpha}(\overline{\Omega})$. For all $v \in C^{1,\beta}(\overline{\Omega})$, the operator T is defined by letting $u = Tv$ be the unique solution in $C^{2,\alpha\beta}(\overline{\Omega})$ of the *linear* Dirichlet problem

$$a^{ij}(x, v, Dv)D_{ij}u + b(x, v, Dv) = 0 \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega.$$

The solvability of the Dirichlet problem

$$\begin{aligned} \mathcal{Q}u &= 0 && \text{in } \Omega, \\ u &= \varphi && \text{on } \partial\Omega, \end{aligned} \tag{2.4.2}$$

in the space $C^{2,\alpha}(\overline{\Omega})$ is thus equivalent to the solvability of the equation $u = Tu$ in the Banach space $B = C^{1,\beta}(\overline{\Omega})$. Moreover, the equation $u = \sigma Tu$ in B is equivalent to the Dirichlet problem

$$\mathcal{Q}_\sigma u = a^{ij}(x, u, Du)D_{ij}u + \sigma b(x, u, Du) = 0 \text{ in } \Omega, \quad u = \sigma\varphi \text{ on } \partial\Omega.$$

By applying Theorem 2.4.1, we have the following criterion for existence.

Theorem 2.4.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ and suppose that $\mathcal{Q}$ is elliptic in $\overline{\Omega}$ with coefficients $a^{ij}, b \in C^\alpha(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N), 0 < \alpha < 1$. Let $\partial\Omega \in C^{2,\alpha}$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$. If for some $\beta > 0$ there exists a constant M, independent of u and σ, such that every $C^{2,\alpha}(\overline{\Omega})$ solution of the Dirichlet problems*

$$\begin{aligned} \mathcal{Q}_\sigma u &= 0 && \text{in } \Omega, \\ u &= \sigma\varphi && \text{on } \partial\Omega, \end{aligned} \qquad 0 \le \sigma \le 1,$$

satisfies

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} < M,$$

then the Dirichlet problem (2.4.2) *is solvable in $C^{2,\alpha}(\overline{\Omega})$.*

Theorem 2.4.2 reduces the solvability of the Dirichlet problem (2.4.2) to the a-priori estimation in the space $C^{1,\beta}(\overline{\Omega})$, for some $\beta > 0$, of the solutions of a related family of problems. In practice it is desirable to break the derivation of the a-priori estimates into four steps:

1. Estimation of $\sup\limits_{\Omega} |u|$;
2. Estimation of $\sup\limits_{\partial\Omega} |Du|$ in terms of $\sup\limits_{\Omega} |u|$;
3. Estimation of $\sup\limits_{\Omega} |Du|$ in terms of $\sup\limits_{\partial\Omega} |Du|$ and $\sup\limits_{\Omega} |u|$;
4. Estimation of $[Du]_{\beta,\Omega}$, for some $\beta > 0$, in terms of $\sup\limits_{\Omega} |Du|$ and $\sup\limits_{\Omega} |u|$.

It is important to observe that the previous results perfectly work when the operator $\mathcal{Q}$ has the special divergence form

$$\mathcal{Q}u = \operatorname{div} \mathcal{A}(Du),$$

as was illustrated in [58, Section 11.3]. Moreover, the geometric conditions on the boundary $\partial\Omega$ play an important role in the solvability of the Dirichlet problem for quasilinear equations, see [58, Chapter 14] for further details.

2.5 Moser iteration technique

The regularity theory is intimately connected with the concept of boundedness of solutions. In this section we describe the main idea of the Moser iteration technique, a performant tool we widely used throughout our work [93]. For a detailed application to eigenvalue problems we refer to [44, 80]. Let Ω be a bounded domain in $\mathbb{R}^N$, $N > 1$, let $p \in]1, +\infty[$ and $a \in L^q(\Omega)$, with $q > N/p$. Consider the following model problem

$$\begin{aligned} -\Delta_p u &= a \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.5.1}$$

We say that a function $u \in W_0^{1,p}(\Omega)$ is a weak solution of (2.5.1) if the following holds

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dx = \int_\Omega a\varphi dx, \quad \forall\, \varphi \in W_0^{1,p}(\Omega).$$

Claim: $u \in L^\infty(\Omega)$. Since $u = u^+ - u^-$, we can suppose without loss of generality that $u \geq 0$. For every $h > 0$ we set $u_h := \min\{u, h\}$ and choose $\varphi = u_h^{\kappa p+1}$ as test function in the equation above, for every $\kappa > 0$. Then we have

$$(\kappa p + 1)\int_{\{x\in\Omega:\, u(x)\leq h\}} |\nabla u|^{p-2}\nabla u \cdot \nabla u u_h^{\kappa p} dx = \int_\Omega a u_h^{\kappa p+1} dx,$$

which simply implies, thanks to Hölder's inequality with exponents q and q', that

$$\frac{\kappa p + 1}{(\kappa+1)^p}\int_\Omega |\nabla u_h^{\kappa+1}|^p dx \leq \|a\|_q \|u_h^{\kappa p+1}\|_{q'}.$$

The Sobolev embedding theorem implies that

$$\frac{1}{c_\Omega^p}\frac{\kappa p + 1}{(\kappa+1)^p}\|u_h^{\kappa+1}\|_{p^*}^p \leq \|a\|_q \|u_h^{\kappa p+1}\|_{q'},$$

with the embedding constant c_Ω. Note that, as $h \to +\infty$, $u_h(x) \to u(x)$ a.e. in Ω. Therefore, applying Fatou's lemma gives

$$\frac{1}{c_\Omega^p}\frac{\kappa p + 1}{(\kappa+1)^p}\|u^{\kappa+1}\|_{p^*}^p \leq \|a\|_q \|u^{\kappa p+1}\|_{q'},$$

that is

$$\|u\|_{(\kappa+1)p^*} \leq c_\Omega^{\frac{1}{\kappa+1}} \left(\frac{\kappa+1}{(\kappa p+1)^{1/p}}\right)^{\frac{1}{(\kappa+1)p}} \|a\|_q^{\frac{1}{(\kappa+1)p}} \|u\|_{(\kappa p+1)q'}^{\frac{\kappa p+1}{(\kappa+1)p}}. \tag{2.5.2}$$

In order to start the Moser iteration technique we need that the norm of u on the left-hand side is greater than the norm of u on the right-hand side. This obviously is satisfied if $(\kappa+1)p^* > (\kappa p+1)q'$ and, in particular, if $(\kappa+1)p^* > (\kappa p+1)\frac{N}{N-p}$, taking into account the hypothesis on q and therefore on q'. Exploiting the calculations, the latter inequality reduces to $p > 1$, which is of course true. We are then ready to start the iteration. Choosing $\kappa_1 > 0$ such that $(\kappa_1 p + 1)q' = p^*$, we inductively construct a sequence $(\kappa_n)_{n\in\mathbb{N}}$ such that $(\kappa_n p+1)q' = (\kappa_{n-1}+1)p^*$. Inserting $\kappa = \kappa_n$ in (2.5.2) gives

$$\|u\|_{(\kappa_n+1)p^*} \leq c_\Omega^{\frac{1}{\kappa_n+1}} \left(\frac{\kappa_n+1}{(\kappa_n p+1)^{1/p}}\right)^{\frac{1}{(\kappa_n+1)p}} \|a\|_q^{\frac{1}{(\kappa_n+1)p}} \|u\|_{(\kappa_{n-1}+1)p^*}^{\frac{\kappa_n p+1}{(\kappa_n+1)p}}, \qquad \forall\, n \in \mathbb{N},$$

which, taking into account that $\kappa_n \simeq \left(\frac{p^*}{pq'}\right)^n$, implies

$$\|u\|_{(\kappa_n+1)p^*} \leq c_\Omega^{\sum_{i=1}^{n}\left(\frac{pq'}{p^*}\right)^i} \prod_{i=1}^{n}\left[\left(\frac{\kappa_i+1}{(\kappa_i p+1)^{1/p}}\right)^{\left(\frac{pq'}{p^*}\right)^i}\right]^{1/p} \|a\|_q^{1/p\sum_{i=1}^{n}\left(\frac{pq'}{p^*}\right)^i} \|u\|_{p^*}.$$

Since it easily follows that $\kappa_n \to +\infty$ as $n \to +\infty$ and that $\frac{pq'}{p^*} < 1$, then there exists a positive constant C such that

$$\|u\|_\infty \leq C\|u\|_{p^*},$$

which in turn entails that $u \in L^\infty(\Omega)$.

2.6 Our results

Let $\Omega \subset \mathbb{R}^N$ with $N > 1$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. We study the boundedness of weak solutions of the problem

$$\begin{aligned} -\operatorname{div}\mathcal{A}(x,u,\nabla u) &= \mathcal{B}(x,u,\nabla u) \qquad && \text{in } \Omega,\\ \mathcal{A}(x,u,\nabla u)\cdot\nu &= \mathcal{C}(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{2.6.1}$$

where $\nu(x)$ denotes the outer unit normal of Ω at $x \in \partial\Omega$, and $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ satisfy suitable p-structure conditions, see hypotheses (H) in Section 2.6.2.

The main goal is to present a-priori bounds for weak solutions of equation (2.6.1), where we allow critical growth to the functions involved both in the domain and on the boundary. The main idea in the proof is based on a modified version of Moser's iteration which in turn is based on the books of Drábek-Kufner-Nicolosi [44] and Struwe [129].

The main novelty of our paper consists in the generality of the assumptions needed to establish the boundedness of weak solutions to (2.6.1). In particular, the assumptions on the nonlinearity $\mathcal{C}$ are rather general allowing critical growth on the boundary. To the best of our knowledge, such a treatment with critical growth even on the boundary has not been studied before.

Recently, Papageorgiou-Rădulescu [102, Proposition 2.8] studied a-priori bounds for problems of the form

$$\begin{aligned} -\operatorname{div} a(\nabla u) &= f_0(x,u) \qquad && \text{in } \Omega,\\ a(\nabla u)\cdot\nu &= -\beta(x)|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned}$$

where $1 < p < \infty$, the function $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is continuous, strictly monotone satisfying certain regularity and growth conditions, the Carathéodory function $f_0\colon \Omega \times \mathbb{R} \to \mathbb{R}$ has critical growth with respect to the second variable and $\beta \in C^{1,\alpha}(\partial\Omega, \mathbb{R}_0^+)$, with $\alpha \in (0,1)$. Note that our setting is more general than those in [102] since we have weaker conditions on a and f_0 and our boundary term is able to have critical growth. The proof of their result is mainly based on a treatment of [51]. Both works use a different technique than the Moser iteration applied in our paper. Moreover, the assumptions on the functions are stronger than ours and no critical growth on the boundary is allowed.

2.6.1 Preliminaries

In this section we present the main preliminaries including a multiplicative inequality estimating the boundary integrals and a result how $L^\infty(\Omega)$-boundedness implies $L^\infty(\partial\Omega)$-boundedness.

Although we have already met the critical exponents on the boundary and in the domain, we recall them here, for the reader's convenience (see [1] for further references):

$$p_* = \begin{cases} \frac{(N-1)p}{N-p} & \text{if } p < N, \\ \text{any } q \in (1,\infty) & \text{if } p \geq N \end{cases} \quad \text{and} \quad p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ \text{any } q \in (1,\infty) & \text{if } p \geq N. \end{cases}$$

The norm of $\mathbb{R}^N$ is denoted by $|\cdot|$ and $\cdot$ stands for the inner product in $\mathbb{R}^N$. By $|\cdot|$ we also denote the Lebesgue measure on $\mathbb{R}^N$.

The following proposition will be useful in our treatment and is based on appropriate embeddings and interpolation results of Besov and Sobolev Slobodeckij spaces (for further details we refer to [135, 136]).

Proposition 2.6.1 (Proposition 2.1 of [150]). *Let $\Omega \subset \mathbb{R}^N$, $N > 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$, let $1 < p < \infty$, and let $\hat{q}$ be such that $p \leq \hat{q} < p_*$. Then, for every $\varepsilon > 0$, there exist constants $\tilde{c}_1 > 0$ and $\tilde{c}_2 > 0$ such that*

$$\|u\|_{\hat{q},\partial\Omega}^p \leq \varepsilon \|u\|_{1,p}^p + \tilde{c}_1 \varepsilon^{-\tilde{c}_2} \|u\|_p^p \qquad \textit{for all } u \in W^{1,p}(\Omega).$$

Proof. Since $\hat{q} < p_*$, we may fix a number $\theta \in (0,1)$ small enough such that

$$\hat{q} \begin{cases} \leq \frac{(N-1)p}{N-p+\theta p} & \text{if } p \leq N \\ < \frac{p-N}{\theta} & \text{if } p > N \end{cases} \quad \text{and} \quad 1-\theta > \frac{1}{p}. \tag{2.6.2}$$

Consider now the continuous embedding

$$B_{p,p}^{1-\theta}(\Omega) = W^{1-\theta,p}(\Omega) \to B_{p,p}^{1-\theta-\frac{1}{p}}(\partial\Omega) = W^{1-\theta-\frac{1}{p},p}(\partial\Omega), \tag{2.6.3}$$

where $B_{p,p}^s$, $s \in (0,1)$, denotes the Besov space which coincides with the Sobolev Slobodeckij space $W^{s,p}$. Note that the embedding (2.6.3) requires only a Lipschitz boundary as $1-\theta < 1$.

From the choice of $\theta \in (0,1)$ and since $p \leq \hat{q}$ (see also (2.6.2)) we get

$$\left(1-\theta-\frac{1}{p}\right)p \begin{cases} < N-1 & \text{if } p \leq N \\ > N-1 & \text{if } p > N. \end{cases}$$

Taking into account the Sobolev embedding theorem for fractional order Sobolev spaces gives

$$W^{1-\theta-\frac{1}{p},p}(\partial\Omega) \to L^{\hat{q}}(\partial\Omega) \tag{2.6.4}$$

for

$$\hat{q} \begin{cases} \leq \frac{(N-1)p}{N-1-(1-\theta-\frac{1}{p})p} = \frac{(N-1)p}{N-p+\theta p} & \text{if } (1-\theta-\frac{1}{p})p < N-1 \\ < \infty & \text{if } (1-\theta-\frac{1}{p})p > N-1. \end{cases}$$

Actually, in case $(1-\theta-\frac{1}{p})p > N-1$ we have the stronger embedding $W^{1-\theta-\frac{1}{p},p}(\partial\Omega) \to C(\partial\Omega)$.

Since $W^{1,p}(\Omega) \subseteq W^{1-\theta,p}(\Omega) \subseteq L^p(\Omega)$ are continuous embedding we may apply real interpolation

$$(L^p(\Omega), W^{1,p}(\Omega))_{1-\theta,p} = W^{1-\theta,p}(\Omega),$$

which implies the estimate

$$\|u\|_{1-\theta,p} \leq \tilde{C}_1 \|u\|_{1,p}^{1-\theta} \|u\|_p^{1-(1-\theta)} \qquad \text{for all } u \in W^{1,p}(\Omega) \tag{2.6.5}$$

with a positive constant $\tilde{C}_1$. Combining (2.6.3)-(2.6.5) and using Young's inequality with $\tilde{\delta} > 0$ results in

$$\begin{aligned} \|u\|_{\hat{q},\partial\Omega}^p &\leq \tilde{C}_2 \tilde{\delta}^{1-\theta} \|u\|_{1,p}^{(1-\theta)p} \tilde{\delta}^{-1+\theta} \|u\|_p^{\theta p} \\ &\leq \tilde{C}_2 \left(\tilde{\delta} \|u\|_{1,p}^p + \tilde{\delta}^{\frac{-1+\theta}{\theta}} \|u\|_p^p \right). \end{aligned}$$

Setting $\tilde{\delta} := \frac{\varepsilon}{\tilde{C}_2}$ with arbitrary $\varepsilon > 0$ provides the desired estimate. $\square$

The next proposition is a standard argument in the application of the Moser iteration, see for example [44].

Proposition 2.6.2. *Let $\Omega \subset \mathbb{R}^N$, $N > 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$. Let $u \in L^p(\Omega)$ with $u \geq 0$ and $1 < p < \infty$ such that*

$$\|u\|_{\alpha_n} \leq C \tag{2.6.6}$$

with a constant $C > 0$ and a sequence $(\alpha_n) \subseteq \mathbb{R}_+$ with $\alpha_n \to \infty$ as $n \to \infty$. Then, $u \in L^\infty(\Omega)$.

Proof. Let us suppose that $u \notin L^\infty(\Omega)$. Then there exist a number $\eta > 0$ and a set A of positive measure in Ω such that $u(x) \geq C + \eta$ for $x \in A$. Then it follows

$$\|u\|_{\alpha_n} \geq \left(\int_A u^{\alpha_n} dx \right)^{\frac{1}{\alpha_n}} \geq (C + \eta) |A|^{\frac{1}{\alpha_n}}.$$

Passing to the limit inferior in the inequality above gives

$$\liminf_{n \to \infty} \|u\|_{\alpha_n} \geq C + \eta,$$

which is a contradiction to (2.6.6). Hence, $u \in L^\infty(\Omega)$. $\square$

Remark 2.6.1. *It is clear that the statement in Proposition 2.6.2 remains true if we replace the domain Ω by its boundary $\partial\Omega$.*

Finally, we state a result that the boundedness of a Sobolev function in $W^{1,p}(\Omega)$ implies the boundedness on the boundary.

Proposition 2.6.3. *Let $\Omega \subset \mathbb{R}^N$, $N > 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $1 < p < \infty$. If $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, then $u \in L^\infty(\partial\Omega)$.*

Proof. By the Sobolev embedding we have

$$\|v\|_{p_*,\partial\Omega} \leq c_{\partial\Omega} \|u\|_{1,p} \quad \text{for all } v \in W^{1,p}(\Omega).$$

Let $\kappa > 1$ and take $v = u^\kappa$ in the inequality above. Note that $v \in W^{1,p}(\Omega)$ since $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. This gives

$$\begin{aligned}\|u\|_{\kappa p_*,\partial\Omega} &\le c_{\partial\Omega}^{\frac{1}{\kappa}} \left[\left(\int_\Omega |\nabla u^\kappa|^p dx \right)^{\frac{1}{\kappa p}} + \left(\int_\Omega |u^k|^p dx \right)^{\frac{1}{\kappa p}} \right] \\ &\le c_{\partial\Omega}^{\frac{1}{\kappa}} \left[\kappa^{\frac{1}{\kappa}} \|u\|_\infty^{1-\frac{1}{\kappa}} \|\nabla u\|_p^{\frac{1}{\kappa}} + \|u\|_\infty |\Omega|^{\frac{1}{\kappa p}} \right].\end{aligned}$$

Letting $\kappa \to \infty$, by applying Proposition 2.6.2 and Remark 2.6.1, we derive

$$\|u\|_{\infty,\partial\Omega} \le 2\|u\|_\infty.$$

□

2.6.2 A-priori bounds via Moser iteration

In this section we state and prove the main result. First, we give the structure conditions on the functions involved in problem (2.6.1).

(H) The functions $\mathcal{A}\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $\mathcal{B}\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, and $\mathcal{C}\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying the following structure conditions:

$$\begin{aligned}
&\text{(H1)} && |\mathcal{A}(x,s,\xi)| \le a_1|\xi|^{p-1} + a_2|s|^{q_1\frac{p-1}{p}} + a_3, && \text{for a.a. } x \in \Omega,\\
&\text{(H2)} && \mathcal{A}(x,s,\xi)\cdot\xi \ge a_4|\xi|^p - a_5|s|^{q_1} - a_6, && \text{for a.a. } x \in \Omega,\\
&\text{(H3)} && |\mathcal{B}(x,s,\xi)| \le b_1|\xi|^{p\frac{q_1-1}{q_1}} + b_2|s|^{q_1-1} + b_3, && \text{for a.a. } x \in \Omega,\\
&\text{(H4)} && |\mathcal{C}(x,s)| \le c_1|s|^{q_2-1} + c_2, && \text{for a.a. } x \in \partial\Omega,
\end{aligned}$$

for all $s \in \mathbb{R}$, for all $\xi \in \mathbb{R}^N$, with positive constants a_i, b_j, c_k ($i \in \{1,\dots,6\}$, $j \in \{1,2,3\}, k \in \{1,2\}$) and fixed numbers p, q_1, q_2 such that

$$1 < p < \infty, \qquad p \le q_1 \le p^*, \qquad p \le q_2 \le p_*$$

A function $u \in W^{1,p}(\Omega)$ is said to be a weak solution of (2.6.1) if

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi dx = \int_\Omega \mathcal{B}(x,u,\nabla u)\varphi dx + \int_{\partial\Omega} \mathcal{C}(x,u)\varphi d\sigma \tag{2.6.7}$$

holds for all test functions $\varphi \in W^{1,p}(\Omega)$. By means of the embeddings $i\colon W^{1,p}(\Omega) \to L^{p^*}(\Omega)$ and $\gamma\colon W^{1,p}(\Omega) \to L^{p_*}(\partial\Omega)$ we see that the definition of a weak solution is well-defined and all integrals in (2.6.7) are finite for $u, \varphi \in W^{1,p}(\Omega)$.

Now we can formulate our main result. Note that we will prove the theorem only for the case $q_1 = p^*$ and $q_2 = p_*$, since the other cases were already obtained in [149, Theorem 4.1] and [150, Theorem 3.1].

Theorem 2.6.1. *Let $\Omega \subset \mathbb{R}^N$, $N > 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$ and let the hypotheses (H) be satisfied. Then, every weak solution $u \in W^{1,p}(\Omega)$ of problem* (2.6.1) *belongs to $L^r(\overline{\Omega})$ for every $r < \infty$. Moreover, $u \in L^\infty(\overline{\Omega})$, that is, $\|u\|_\infty \le M$, where M is a constant which depends on the given data and on u.*

Proof. Let $u \in W^{1,p}(\Omega)$ be a weak solution of problem (2.6.1). Reasoning as in Section 2.5 we can suppose that $u \ge 0$. Furthermore, we will denote positive constants with M_i and if the constant depends on the parameter κ we write $M_i(\kappa)$ for $i = 1, 2, \dots$.

Let $h > 0$ and set $u_h := \min\{u, h\}$. Then we choose $\varphi = u u_h^{\kappa p}$ with $\kappa > 0$ as test function in (2.6.7). This gives

$$\begin{aligned}\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot \nabla u u_h^{\kappa p} + \kappa p \int_{\{x\in\Omega\colon u(x)\le h\}} \mathcal{A}(x,u,\nabla u)\cdot \nabla u u_h^{\kappa p} dx&\\ = \int_\Omega \mathcal{B}(x,u,\nabla u) u u_h^{\kappa p} dx + \int_{\partial\Omega} \mathcal{C}(x,u) u u_h^{\kappa p} d\sigma.&\end{aligned} \tag{2.6.8}$$

Applying (H2) to the first term of the left-hand side of (2.6.8) yields

$$\begin{aligned}&\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot \nabla u u_h^{\kappa p}\\ &\ge \int_\Omega \left[a_4|\nabla u|^p - a_5 u^{p^*} - a_6\right] u_h^{\kappa p} dx\\ &\ge a_4 \int_\Omega |\nabla u|^p u_h^{\kappa p} dx - (a_5+a_6)\int_\Omega u^{p^*} u_h^{\kappa p} dx - a_6|\Omega|,\end{aligned}$$

respectively to the second term on the left-hand side

$$\begin{aligned}&\kappa p \int_{\{x\in\Omega\colon u(x)\le h\}} \mathcal{A}(x,u,\nabla u)\cdot \nabla u u_h^{\kappa p} dx\\ &\ge \kappa p \int_{\{x\in\Omega\colon u(x)\le h\}} \left[a_4|\nabla u|^p - a_5 u^{p^*} - a_6\right] u_h^{\kappa p} dx\\ &\ge a_4 \kappa p \int_{\{x\in\Omega\colon u(x)\le h\}} |\nabla u|^p u_h^{\kappa p} dx - \kappa p(a_5+a_6)\int_\Omega u^{p^*} u_h^{\kappa p} dx - a_6 \kappa p|\Omega|.\end{aligned}$$

By means of (H3) combined with Young's inequality with $\varepsilon_1 > 0$, the first term on the right-hand side of (2.6.8) can be estimated through

$$\begin{aligned}&\int_\Omega \mathcal{B}(x,u,\nabla u) u u_h^{\kappa p} dx\\ &\le b_1 \int_\Omega \varepsilon_1^{\frac{p^*-1}{p^*}} |\nabla u|^{p\frac{p^*-1}{p^*}} u_h^{\kappa p\frac{p^*-1}{p^*}} \varepsilon_1^{-\frac{p^*-1}{p^*}} u_h^{\kappa p(1-\frac{p^*-1}{p^*})} u dx\\ &\qquad + (b_2+b_3)\int_\Omega u^{p^*} u_h^{\kappa p} dx + b_3|\Omega|\\ &\le \varepsilon_1 b_1 \int_\Omega |\nabla u|^p u_h^{\kappa p} dx + \left(b_1 \varepsilon_1^{-(p^*-1)} + b_2 + b_3\right)\int_\Omega u^{p^*} u_h^{\kappa p} dx + b_3|\Omega|.\end{aligned} \tag{2.6.9}$$

Finally, the boundary term can be estimated via (H4). This leads to

$$\begin{aligned}\int_{\partial\Omega} \mathcal{C}(x,u) u u_h^{\kappa p} d\sigma &\le \int_{\partial\Omega} \left(c_1 u^{p_*-1} + c_2\right) u u_h^{\kappa p} d\sigma\\ &\le (c_1+c_2)\int_{\partial\Omega} u^{p_*} u_h^{\kappa p} d\sigma + c_2|\partial\Omega|.\end{aligned} \tag{2.6.10}$$

Taking into account all these inequalities, equation (2.6.8) can be written as

$$\begin{aligned}a_4\Big(\int_\Omega |\nabla u|^p u_h^{\kappa p} dx + \kappa p \int_{\{x\in\Omega\colon u(x)\le h\}} |\nabla u|^p u_h^{\kappa p} dx\Big) &\le \varepsilon_1 b_1 \int_\Omega |\nabla u|^p u_h^{\kappa p} dx\\ &+ \left[(\kappa p+1)(a_5+a_6) + b_1\varepsilon_1^{-(p^*-1)} + b_2 + b_3\right]\int_\Omega u^{p^*} u_h^{\kappa p} dx\\ &+ (c_1+c_2)\int_{\partial\Omega} u^{p_*} u_h^{\kappa p} d\sigma + ((\kappa p+1)a_6 + b_3)|\Omega| + c_2|\partial\Omega|.\end{aligned}$$

Choosing $\varepsilon_1 = \frac{a_4}{2b_1}$ then it easily follows that

$$\begin{aligned}
&\frac{a_4}{2}\frac{\kappa p+1}{(\kappa+1)^p}\int_\Omega |\nabla(uu_h^\kappa)|^p dx\\
&\qquad\qquad \leq [(\kappa p+1)(a_5+a_6)+\varepsilon_1^{-(p^*-1)}b_1+b_2+b_3]\int_\Omega u^{p^*}u_h^{\kappa p}dx \qquad (2.6.11)\\
&\qquad\qquad + (c_1+c_2)\int_{\partial\Omega} u^{p_*}u_h^{\kappa p}d\sigma + ((\kappa p+1)a_6+b_3)|\Omega| + c_2|\partial\Omega|.
\end{aligned}$$

Dividing by a_4, summarizing the constants and adding on both sides of (2.6.11) the nonnegative term $\frac{\kappa p+1}{(\kappa+1)^p}\|uu_h^\kappa\|_p^p$ gives

$$\begin{aligned}
&\frac{\kappa p+1}{(\kappa+1)^p}\|uu_h^\kappa\|_{1,p}^p\\
&\leq \frac{\kappa p+1}{(\kappa+1)^p}\|uu_h^\kappa\|_p^p + M_1(\kappa p+1)\int_\Omega u^{p^*}u_h^{\kappa p}dx + M_2\int_{\partial\Omega} u^{p_*}u_h^{\kappa p}d\sigma + M_3.
\end{aligned} \qquad (2.6.12)$$

Part I: $u \in L^r(\overline{\Omega})$ for any finite r.

Let us now estimate the terms on the right-hand side involving the critical exponents. We set $a := u^{p^*-p}$ and $b := u^{p_*-p}$. Moreover, let $L > 0$ and $G > 0$. Then, by using Hölder's inequality and the Sobolev embeddings for p^* and p_* we get

$$\begin{aligned}
&\int_\Omega u^{p^*}u_h^{\kappa p}dx = \int_\Omega u^{p^*-p}(uu_h^\kappa)^p dx\\
&= \int_{\{x\in\Omega:\, a(x)\leq L\}} a(uu_h^\kappa)^p dx + \int_{\{x\in\Omega:\, a(x)>L\}} a(uu_h^\kappa)^p dx\\
&\leq L\int_{\{x\in\Omega:\, a(x)\leq L\}} (uu_h^\kappa)^p dx\\
&\quad + \left(\int_{\{x\in\Omega:\, a(x)>L\}} a^{\frac{p^*}{p^*-p}} dx\right)^{\frac{p^*-p}{p^*}} \left(\int_\Omega (uu_h^\kappa)^{p^*}dx\right)^{\frac{p}{p^*}}\\
&\leq L\|uu_h^\kappa\|_p^p + \left(\int_{\{x\in\Omega:\, a(x)>L\}} a^{\frac{p^*}{p^*-p}} dx\right)^{\frac{p^*-p}{p^*}} c_\Omega^p\|uu_h^\kappa\|_{1,p}^p
\end{aligned} \qquad (2.6.13)$$

and

$$\begin{aligned}
&\int_{\partial\Omega} u^{p_*}u_h^{\kappa p}d\sigma = \int_{\partial\Omega} u^{p_*-p}(uu_h^\kappa)^p d\sigma\\
&= \int_{\{x\in\partial\Omega:\, b(x)\leq G\}} b(uu_h^\kappa)^p d\sigma + \int_{\{x\in\partial\Omega:\, b(x)>G\}} b(uu_h^\kappa)^p d\sigma\\
&\leq G\int_{\{x\in\partial\Omega:\, b(x)\leq G\}} (uu_h^\kappa)^p d\sigma\\
&\quad + \left(\int_{\{x\in\partial\Omega:\, b(x)>G\}} b^{\frac{p_*}{p_*-p}} d\sigma\right)^{\frac{p_*-p}{p_*}} \left(\int_{\partial\Omega} (uu_h^\kappa)^{p_*}d\sigma\right)^{\frac{p}{p_*}}\\
&\leq G\|uu_h^\kappa\|_{p,\partial\Omega}^p + \left(\int_{\{x\in\partial\Omega:\, b(x)>G\}} b^{\frac{p_*}{p_*-p}} d\sigma\right)^{\frac{p_*-p}{p_*}} c_{\partial\Omega}^p\|uu_h^\kappa\|_{1,p}^p
\end{aligned} \qquad (2.6.14)$$

with the embedding constants c_Ω and $c_{\partial\Omega}$. Note that

$$\begin{aligned}
H(L) &:= \left(\int_{\{x\in\Omega:\, a(x)>L\}} a^{\frac{p^*}{p^*-p}} dx\right)^{\frac{p^*-p}{p^*}} \to 0 \quad \text{as } L\to\infty,\\
K(G) &:= \left(\int_{\{x\in\partial\Omega:\, b(x)>G\}} b^{\frac{p_*}{p_*-p}} d\sigma\right)^{\frac{p_*-p}{p_*}} \to 0 \quad \text{as } G\to\infty.
\end{aligned} \qquad (2.6.15)$$

Combining (2.6.12)-(2.6.15) finally yields

$$\begin{aligned} &\frac{\kappa p+1}{(\kappa+1)^p}\|uu_h^\kappa\|_{1,p}^p \\ &\leq \left[\frac{\kappa p+1}{(\kappa+1)^p}+M_1(\kappa p+1)L\right]\|uu_h^\kappa\|_p^p+M_1(\kappa p+1)H(L)c_\Omega^p\|uu_h^\kappa\|_{1,p}^p \\ &\qquad +M_2G\|uu_h^\kappa\|_{p,\partial\Omega}^p+M_2K(G)c_{\partial\Omega}^p\|uu_h^\kappa\|_{1,p}^p+M_3. \end{aligned} \tag{2.6.16}$$

Now we choose $L=L(\kappa,u)>0$ and $G=G(\kappa,u)>0$ such that

$$M_1(\kappa p+1)H(L)c_\Omega^p=\frac{\kappa p+1}{4(\kappa+1)^p},\qquad M_2K(G)c_{\partial\Omega}^p=\frac{\kappa p+1}{4(\kappa+1)^p}.$$

Then, (2.6.16) becomes

$$\begin{aligned} &\frac{\kappa p+1}{2(\kappa+1)^p}\|uu_h^\kappa\|_{1,p}^p \\ &\leq \left[\frac{\kappa p+1}{(\kappa+1)^p}+M_1(\kappa p+1)L(\kappa,u)\right]\|uu_h^\kappa\|_p^p+M_2G(\kappa,u)\|uu_h^\kappa\|_{p,\partial\Omega}^p+M_3, \end{aligned} \tag{2.6.17}$$

where $L(\kappa,u)$ and $G(\kappa,u)$ depend on κ and on the solution itself.

We can use Proposition 2.6.1 to estimate the remaining boundary term in the form of

$$\|uu_h^\kappa\|_{p,\partial\Omega}^p\leq\varepsilon_2\|uu_h^\kappa\|_{1,p}^p+\tilde{c}_1\varepsilon_2^{-\tilde{c}_2}\|uu_h^\kappa\|_p^p. \tag{2.6.18}$$

Choosing $\varepsilon_2=\frac{1}{M_2G(\kappa)}\frac{\kappa p+1}{4(\kappa+1)^p}$ and applying (2.6.18) to (2.6.17) give

$$\begin{aligned} &\frac{\kappa p+1}{4(\kappa+1)^p}\|uu_h^\kappa\|_{1,p}^p \\ &\leq \left[\frac{\kappa p+1}{(\kappa+1)^p}+M_1(\kappa p+1)L(\kappa,u)+M_2G(\kappa,u)\tilde{c}_1\varepsilon_2^{-\tilde{c}_2}\right]\|uu_h^\kappa\|_p^p+M_3. \end{aligned} \tag{2.6.19}$$

Inequality (2.6.19) can be rewritten as

$$\|uu_h^\kappa\|_{1,p}^p\leq M_4(\kappa,u)\left[\|uu_h^\kappa\|_p^p+1\right] \tag{2.6.20}$$

with a constant $M_4(\kappa,u)$ depending on κ and u. We may apply the Sobolev embedding on the left-hand side of (2.6.20) which leads to

$$\|uu_h^\kappa\|_{p^*}^p\leq c_\Omega^p\|uu_h^\kappa\|_{1,p}^p\leq M_5(\kappa,u)\left[\|uu_h^\kappa\|_p^p+1\right].$$

Since, as $h\to+\infty$, $u_h(x)\to u(x)$ a.e., then applying Fatou's lemma gives

$$\|u\|_{(\kappa+1)p^*}=\|u^{\kappa+1}\|_{p^*}^{\frac{1}{\kappa+1}}\leq M_5(\kappa,u)\left[\|u^{\kappa+1}\|_p^p+1\right]^{\frac{1}{(\kappa+1)p}}.$$

Now we can start with the typical bootstrap argument. Choosing κ such that

$$\begin{aligned} &\kappa_1:(\kappa_1+1)p=p^*, \\ &\kappa_2:(\kappa_2+1)p=(\kappa_1+1)p^*, \\ &\kappa_3:(\kappa_3+1)p=(\kappa_2+1)p^*, \\ &\qquad\vdots\qquad\qquad\qquad\vdots\quad, \end{aligned}$$

we see that

$$\|u\|_{(\kappa+1)p^*} \leq M_6(\kappa, u) \tag{2.6.21}$$

for any finite number κ, where $M_6(\kappa, u)$ is a positive constant depending on κ and on the solution u. Thus, $u \in L^r(\Omega)$ for any $r \in (1, \infty)$.

We now want to prove that $u \in L^r(\partial\Omega)$ for any finite r. Let us repeat inequality (2.6.17) which says

$$\begin{aligned}
&\frac{\kappa p+1}{2(\kappa+1)^p}\|uu_h^\kappa\|_{1,p}^p \\
&\leq \left[\frac{\kappa p+1}{(\kappa+1)^p} + M_7(\kappa p+1)L(\kappa, u)\right]\|uu_h^\kappa\|_p^p + M_8 G(\kappa, u)\|uu_h^\kappa\|_{p,\partial\Omega}^p + M_9.
\end{aligned} \tag{2.6.22}$$

Taking into account (2.6.21) we can write (2.6.22) in the form

$$\|uu_h^\kappa\|_{1,p}^p \leq M_{10}(\kappa, u)\left[\|uu_h^\kappa\|_{p,\partial\Omega}^p + 1\right]. \tag{2.6.23}$$

We may apply the Sobolev embedding for the boundary on the left-hand side of (2.6.23), which gives

$$\|uu_h^\kappa\|_{p_*,\partial\Omega} \leq c_{\partial\Omega}\|uu_h^\kappa\|_{1,p} \leq M_{11}(\kappa, u)\left[\|uu_h^\kappa\|_{p,\partial\Omega}^p + 1\right].$$

Since $u_h(x) \to u(x)$ a. e. as $h \to +\infty$, applying again Fatou's lemma we then have

$$\|u\|_{(\kappa+1)p_*,\partial\Omega} = \|u^{\kappa+1}\|_{p_*,\partial\Omega}^{\frac{1}{\kappa+1}} \leq M_{12}(\kappa, u)\left[\|u^{\kappa+1}\|_{p,\partial\Omega}^p + 1\right]^{\frac{1}{(\kappa+1)p}}.$$

As before we proceed with a bootstrap argument and choose a sequence (κ_n) in the following way

$$\begin{aligned}
\kappa_1 &: (\kappa_1+1)p = p_*, \\
\kappa_2 &: (\kappa_2+1)p = (\kappa_1+1)p_*, \\
\kappa_3 &: (\kappa_3+1)p = (\kappa_2+1)p_*, \\
&\vdots \qquad\qquad\qquad \vdots \quad .
\end{aligned}$$

We obtain

$$\|u\|_{(\kappa+1)p_*,\partial\Omega} \leq M_{13}(\kappa, u) \tag{2.6.24}$$

for any finite number κ, where $M_{13}(\kappa, u)$ is a positive constant depending on κ and on the solution itself. Thus, $u \in L^r(\partial\Omega)$ for any $r \in (1, \infty)$, and therefore $u \in L^r(\overline{\Omega})$ for any finite $r \in (1, \infty)$.

Part II: $u \in L^\infty(\overline{\Omega})$**.**

Let us recall inequality (2.6.12) which says

$$\begin{aligned}
&\frac{\kappa p+1}{(\kappa+1)^p}\|uu_h^\kappa\|_{1,p}^p \\
&\leq \frac{\kappa p+1}{(\kappa+1)^p}\|uu_h^\kappa\|_p^p + M_{14}(\kappa p+1)\int_\Omega u^{p^*}u_h^{\kappa p}dx \\
&+ M_{15}\int_{\partial\Omega} u^{p_*}u_h^{\kappa p}d\sigma + M_{16}.
\end{aligned} \tag{2.6.25}$$

Let us fix numbers $\tilde{q}_1 \in (p, p^*)$ and $\tilde{q}_2 \in (p, p_*)$. Then, by applying Hölder's inequality, (2.6.21) and (2.6.24), we derive for the several terms on the right-hand side of (2.6.25) the following estimates

$$\begin{aligned}
\|uu_h^\kappa\|_p^p &\leq |\Omega|^{\frac{\tilde{q}_1-p}{\tilde{q}_1}} \left(\int_\Omega (uu_h^\kappa)^{\tilde{q}_1} dx\right)^{\frac{p}{\tilde{q}_1}} \leq M_{17} \|uu_h^\kappa\|_{\tilde{q}_1}^p, \\
\int_\Omega u^{p^*} u_h^{\kappa p} dx &= \int_\Omega u^{p^*-p} (uu_h^\kappa)^p dx \\
&\leq \left(\int_\Omega u^{\frac{p^*-p}{\tilde{q}_1-p}\tilde{q}_1} dx\right)^{\frac{\tilde{q}_1-p}{\tilde{q}_1}} \left(\int_\Omega (uu_h^\kappa)^{\tilde{q}_1}\right)^{\frac{p}{\tilde{q}_1}} \\
&\leq M_{18} \|uu_h^\kappa\|_{\tilde{q}_1}^p, \\
\int_{\partial\Omega} u^{p_*} u_h^{\kappa p} d\sigma &= \int_{\partial\Omega} u^{p_*-p} (uu_h^\kappa)^p \\
&\leq \left(\int_{\partial\Omega} u^{\frac{p_*-p}{\tilde{q}_2-p}\tilde{q}_2} d\sigma\right)^{\frac{\tilde{q}_2-p}{\tilde{q}_2}} \left(\int_{\partial\Omega} (uu_h^\kappa)^{\tilde{q}_2} d\sigma\right)^{\frac{p}{\tilde{q}_2}} \\
&\leq M_{19} \|uu_h^\kappa\|_{\tilde{q}_2,\partial\Omega}^p.
\end{aligned} \tag{2.6.26}$$

Note that M_{18}, M_{19} are finite taking into account Part I. Moreover we see from the calculations above that

$$M_{18} = M_{18}\left(\|u\|_{\frac{p^*-p}{\tilde{q}_1-p}\tilde{q}_1}\right) \quad \text{and} \quad M_{19} = M_{19}\left(\|u\|_{\frac{p_*-p}{\tilde{q}_2-p}\tilde{q}_2}\right). \tag{2.6.27}$$

Using (2.6.26) to (2.6.25) leads to

$$\frac{\kappa p+1}{(\kappa+1)^p} \|uu_h^\kappa\|_{1,p}^p \leq M_{20}(\kappa p+1)\|uu_h^\kappa\|_{\tilde{q}_1}^p + M_{21}\|uu_h^\kappa\|_{\tilde{q}_2,\partial\Omega}^p + M_{22}. \tag{2.6.28}$$

As before, we can estimate the boundary term via Proposition 2.6.1 and then use Hölder's inequality as made in the first line of (2.6.26). This gives

$$\begin{aligned}
\|uu_h^\kappa\|_{\tilde{q}_2,\partial\Omega}^p &\leq \varepsilon_3 \|uu_h^\kappa\|_{1,p}^p + \tilde{c}_1 \varepsilon_3^{-\tilde{c}_2} \|uu_h^\kappa\|_p^p \\
&\leq \varepsilon_3 \|uu_h^\kappa\|_{1,p}^p + \tilde{c}_1 \varepsilon_3^{-\tilde{c}_2} M_{23} \|uu_h^\kappa\|_{\tilde{q}_1}^p.
\end{aligned} \tag{2.6.29}$$

Now we choose $\varepsilon_3 = \frac{\kappa p+1}{2M_{21}(\kappa+1)^p}$ and apply (2.6.29) in (2.6.28) to obtain

$$\frac{\kappa p+1}{2(\kappa+1)^p} \|uu_h^\kappa\|_{1,p}^p \leq \left(M_{20}(\kappa p+1) + \tilde{c}_1 \varepsilon_3^{-\tilde{c}_2} M_{21} M_{23}\right) \|uu_h^\kappa\|_{\tilde{q}_1}^p + M_{22}. \tag{2.6.30}$$

Inequality (2.6.30) can be rewritten in the form

$$\|uu_h^\kappa\|_{1,p}^p \leq M_{24}\left((\kappa+1)^p\right)^{M_{25}} \left[\|uu_h^\kappa\|_{\tilde{q}_1}^p + 1\right]. \tag{2.6.31}$$

In order so see this, note that

$$\begin{aligned}
&\frac{2(\kappa+1)^p}{\kappa p+1}\left(M_{20}(\kappa p+1) + \tilde{c}_1 \varepsilon_3^{-\tilde{c}_2} M_{21} M_{23}\right) \\
&= 2(\kappa+1)^p \left(M_{20} + \tilde{c}_1 \left(\frac{2M_{21}(\kappa+1)^p}{\kappa p+1}\right)^{\tilde{c}_2} \frac{1}{\kappa p+1} M_{21} M_{23}\right) \\
&\leq M_{24}\left((\kappa+1)^p\right)^{M_{25}}.
\end{aligned}$$

Now we may apply the Sobolev embedding and the Fatou's lemma on the left-hand side of (2.6.31) to get

$$\begin{aligned} \|u\|_{(\kappa+1)p^*} &= \|u^{\kappa+1}\|_{p^*}^{\frac{1}{\kappa+1}} \leq c_\Omega^{\frac{1}{\kappa+1}} \|u^{\kappa+1}\|_{1,p}^{\frac{1}{\kappa+1}} \\ &\leq M_{26}^{\frac{1}{\kappa+1}} \left((\kappa+1)^{M_{25}}\right)^{\frac{1}{\kappa+1}} \left[\|u^{\kappa+1}\|_{\tilde{q}_1}^p + 1\right]^{\frac{1}{(\kappa+1)p}}. \end{aligned} \tag{2.6.32}$$

Observe that

$$\left((\kappa+1)^{M_{25}}\right)^{\frac{1}{\sqrt{\kappa+1}}} \geq 1 \qquad \text{and} \qquad \lim_{\kappa\to\infty} \left((\kappa+1)^{M_{25}}\right)^{\frac{1}{\sqrt{\kappa+1}}} = 1.$$

Hence, we find a constant $M_{27} > 1$ such that

$$\left((\kappa+1)^{M_{25}}\right)^{\frac{1}{\kappa+1}} \leq M_{27}^{\frac{1}{\sqrt{\kappa+1}}}. \tag{2.6.33}$$

From (2.6.32) and (2.6.33) we derive

$$\|u\|_{(\kappa+1)p^*} \leq M_{26}^{\frac{1}{\kappa+1}} M_{27}^{\frac{1}{\sqrt{\kappa+1}}} \left[\|u^{\kappa+1}\|_{\tilde{q}_1}^p + 1\right]^{\frac{1}{(\kappa+1)p}}. \tag{2.6.34}$$

Now we are ready to prove the uniform boundedness with respect to κ. To this end, suppose there is a sequence $\kappa_n \to \infty$ such that

$$\|u^{\kappa_n+1}\|_{\tilde{q}_1}^p \leq 1,$$

which is equivalent to

$$\|u\|_{(\kappa_n+1)\tilde{q}_1} \leq 1,$$

then Proposition 2.6.2 implies that $\|u\|_\infty < \infty$.

In the opposite case there exists a number $\kappa_0 > 0$ such that

$$\|u^{\kappa+1}\|_{\tilde{q}_1}^p > 1 \qquad \text{for any } \kappa \geq \kappa_0. \tag{2.6.35}$$

Combining (2.6.34) and (2.6.35) yields

$$\|u\|_{(\kappa+1)p^*} \leq M_{26}^{\frac{1}{\kappa+1}} M_{27}^{\frac{1}{\sqrt{\kappa+1}}} \left[2\|u^{\kappa+1}\|_{\tilde{q}_1}^p\right]^{\frac{1}{(\kappa+1)p}} \leq M_{28}^{\frac{1}{\kappa+1}} M_{27}^{\frac{1}{\sqrt{\kappa+1}}} \|u\|_{(\kappa+1)\tilde{q}_1}. \tag{2.6.36}$$

Applying again the bootstrap argument we define a sequence (κ_n) such that

$$\begin{aligned} \kappa_1 &: (\kappa_1+1)\tilde{q}_1 = (\kappa_0+1)p^*, \\ \kappa_2 &: (\kappa_2+1)\tilde{q}_1 = (\kappa_1+1)p^*, \\ \kappa_3 &: (\kappa_3+1)\tilde{q}_1 = (\kappa_2+1)p^*, \\ &\vdots \qquad\qquad\qquad \vdots \quad . \end{aligned} \tag{2.6.37}$$

By induction, from (2.6.36) and (2.6.37) we obtain

$$\|u\|_{(\kappa_n+1)p^*} \leq M_{28}^{\frac{1}{\kappa_n+1}} M_{27}^{\frac{1}{\sqrt{\kappa_n+1}}} \|u\|_{(\kappa_n+1)\tilde{q}_1} = M_{28}^{\frac{1}{\kappa_n+1}} M_{27}^{\frac{1}{\sqrt{\kappa_n+1}}} \|u\|_{(\kappa_{n-1}+1)p^*}$$

for any $n \in \mathbb{N}$, where the sequence (κ_n) is chosen in such a way that $(\kappa_n+1) = (\kappa_0+1)\left(\frac{p^*}{\tilde{q}_1}\right)^n$. Following this we see that

$$\|u\|_{(\kappa_n+1)p^*} \leq M_{28}^{\sum\limits_{i=1}^n \frac{1}{\kappa_i+1}} M_{27}^{\sum\limits_{i=1}^n \frac{1}{\sqrt{\kappa_i+1}}} \|u\|_{(\kappa_0+1)p^*}$$

with $(\kappa_n+1)p^* \to \infty$ as $n \to \infty$. Since $\frac{1}{\kappa_i+1} = \frac{1}{\kappa_0+1}\left(\frac{\tilde{q}_1}{p^*}\right)^i$ and $\frac{\tilde{q}_1}{p^*} < 1$, there is a constant $M_{29} > 0$ such that

$$\|u\|_{(\kappa_n+1)p^*} \leq M_{29}\|u\|_{(k_0+1)p^*} < \infty,$$

where the finiteness of the right-hand side follows from Part I. Now we may apply Proposition 2.6.2 to conclude that $u \in L^\infty(\Omega)$, that is, there exists $M > 0$, which depends on the given data and on u, such that $\|u\|_\infty \leq M$.

Moreover, taking into account Proposition 2.6.3, we have that $u \in L^\infty(\partial\Omega)$, and so $u \in L^\infty(\overline{\Omega})$. The proof is thus complete. □

Remark 2.6.2. *It is clear that hypothesis (H1) is not needed in the proof of Theorem 2.6.1, but it is necessary to have a well-defined definition of a weak solution.*

Remark 2.6.3. *Since problem* (2.6.1) *involves functions that can exhibit a critical growth, one cannot expect to find a constant M which depends in an explicit way on natural norms such as $\|u\|_{p^*}$ or $\|u\|_{p_*,\partial\Omega}$. But, if one searches for a dependence on norms that are greater than the critical ones, then a possible dependence is given on the norms $\|u\|_{\frac{p^*-p}{\tilde{q}_1-p}\tilde{q}_1}$ as well as $\|u\|_{\frac{p_*-p}{\tilde{q}_2-p}\tilde{q}_2,\partial\Omega}$, where $\tilde{q}_1 \in (p,p^*)$ and $\tilde{q}_2 \in (p,p_*)$, as seen in the proof of Theorem 2.6.1, cf.* (2.6.27).

2.6.3 Some regularity results

Based on the results of Theorem 2.6.1, we obtain regularity results for solutions of problem (2.6.1). For simplification we drop the s-dependence of the operator. To this end, let $\vartheta \in C^1(0,\infty)$ be a function such that

$$0 < a_1 \leq \frac{t\vartheta'(t)}{\vartheta(t)} \leq a_2 \quad \text{and} \quad a_3t^{p-1} \leq \vartheta(t) \leq a_4\left(1+t^{p-1}\right) \tag{2.6.38}$$

for all $t > 0$, with some constants $a_i > 0$, $i \in \{1,2,3,4\}$ and for $1 < p < \infty$. The hypotheses on $\mathcal{A}\colon \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ read as follows.

H($\mathcal{A}$): $\mathcal{A}(x,\xi) = \mathcal{A}_0(x,|\xi|)\xi$ with $\mathcal{A}_0 \in C(\overline{\Omega} \times \mathbb{R}_0^+)$ for all $\xi \in \mathbb{R}^N$ and with $\mathcal{A}_0(x,t) > 0$ for all $x \in \overline{\Omega}$ and for all $t > 0$. Moreover,

(i) $\mathcal{A}_0 \in C^1(\overline{\Omega} \times (0,\infty))$, $t \to t\mathcal{A}_0(x,t)$ is strictly increasing in $(0,\infty)$, $\lim_{t\to 0^+} t\mathcal{A}_0(x,t) = 0$ for all $x \in \overline{\Omega}$ and

$$\lim_{t\to 0^+}\frac{t\mathcal{A}_0'(x,t)}{\mathcal{A}_0(x,t)} = c > -1 \quad \text{for all } x \in \overline{\Omega};$$

(ii) $|\nabla_\xi \mathcal{A}(x,\xi)| \leq a_5\dfrac{\vartheta(|\xi|)}{|\xi|}$ for all $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and for some $a_5 > 0$;

(iii) $\nabla_\xi \mathcal{A}(x,\xi)y \cdot y \geq \dfrac{\vartheta(|\xi|)}{|\xi|}|y|^2$ for all $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and for all $y \in \mathbb{R}^N$.

Remark 2.6.4. *We chose the special structure in H($\mathcal{A}$) to apply the nonlinear regularity theory, which is mainly based on the results of Lieberman [84] and Pucci-Serrin [112]. If we set*

$$G_0(x,t) = \int_0^t \mathcal{A}_0(x,s)sds,$$

then $G_0 \in C^1(\overline{\Omega} \times \mathbb{R}^+)$ and the function $G_0(x, \cdot)$ is increasing and strictly convex for all $x \in \overline{\Omega}$. We set $G(x, \xi) = G_0(x, |\xi|)$ for all $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^N$ and obtain that $G \in C^1(\overline{\Omega} \times \mathbb{R}^N)$ and that the function $\xi \to G(x, \xi)$ is convex. Moreover, we easily derive that

$$\nabla_\xi G(x, \xi) = (G_0)'_t(x, |\xi|)\frac{\xi}{|\xi|} = \mathcal{A}_0(x, |\xi|)\xi = \mathcal{A}(x, \xi)$$

for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and $\nabla_\xi G(x, 0) = 0$. So, $G(x, \cdot)$ is the primitive of $\mathcal{A}(x, \cdot)$. This fact, the convexity of $G(x, \cdot)$ and since $G(x, 0) = 0$ for all $x \in \overline{\Omega}$ imply that

$$G(x, \xi) \leq \mathcal{A}(x, \xi) \cdot \xi \quad \text{for all } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^N. \tag{2.6.39}$$

The next lemma summarizes the main properties of $\mathcal{A}\colon \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$. The result is an easy consequence of (2.6.38) and the hypotheses H($\mathcal{A}$).

Lemma 2.6.1. *If hypotheses H($\mathcal{A}$) are satisfied, then the following hold:*

(i) $\mathcal{A} \in C(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R}^N)$ and the map $\xi \to \mathcal{A}(x, \xi)$ is continuous and strictly monotone (hence, maximal monotone) for all $x \in \overline{\Omega}$;

(ii) $|\mathcal{A}(x, \xi)| \leq a_6 \left(1 + |\xi|^{p-1}\right)$ for all $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N$ and for some $a_6 > 0$;

(iii) $\mathcal{A}(x, \xi) \cdot \xi \geq \frac{a_3}{p-1}|\xi|^p$ for all $x \in \overline{\Omega}$ and for all $\xi \in \mathbb{R}^N$.

From this lemma along with (2.6.39) we easily deduce the following growth estimates for the primitive $G(x, \cdot)$.

Corollary 2.6.1. *If hypotheses H($\mathcal{A}$) hold, then*

$$\frac{a_3}{p(p-1)}|\xi|^p \leq G(x, \xi) \leq a_7 \left(1 + |\xi|^p\right)$$

for all $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N$ and for some $a_7 > 0$.

Let $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), \varphi \rangle = \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi dx \quad \text{for all } u, \varphi \in W^{1,p}(\Omega). \tag{2.6.40}$$

The next proposition summarizes the main properties of this operator, see [52].

Proposition 2.6.4. *Let the hypotheses H($\mathcal{A}$) be satisfied and let $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the map defined in (2.6.40). Then, A is bounded, continuous, monotone (hence maximal monotone) and of type (S_+).*

Let us state some operators which fit in our setting and which are of much interest.

Example 2.6.1. *For simplicity, we drop the x-dependence of the operator $\mathcal{A}$. The following maps satisfy hypotheses H($\mathcal{A}$):*

(i) Let $\mathcal{A}(\xi) = |\xi|^{p-2}\xi$ with $1 < p < \infty$. This map corresponds to the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \text{for all } u \in W^{1,p}(\Omega).$$

The potential is $G(\xi) = \frac{1}{p}|\xi|^p$ for all $\xi \in \mathbb{R}^N$.

(ii) *The function* $\mathcal{A}(\xi) = |\xi|^{p-2}\xi + \mu|\xi|^{q-2}\xi$ *with* $1 < q < p < \infty$ *and* $\mu > 0$ *compares with the* (p,q)*-differential operator defined by* $\Delta_p u + \mu \Delta_q u$ *for all* $u \in W^{1,p}(\Omega)$*. The potential is* $G(\xi) = \frac{1}{p}|\xi|^p + \frac{\mu}{q}|\xi|^q$ *for all* $\xi \in \mathbb{R}^N$*.*

(iii) *If* $\mathcal{A}(\xi) = \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} \xi$ *with* $1 < p < \infty$*, then this map represents the generalized* p*-mean curvature differential operator defined by*

$$\operatorname{div}\left[(1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u\right] \qquad \text{for all } u \in W^{1,p}(\Omega).$$

The potential is $G(\xi) = \frac{1}{p}\left[(1 + |\xi|^2)^{\frac{p}{2}} - 1\right]$ *for all* $\xi \in \mathbb{R}^N$*.*

Let us write hypotheses (H) without the structure conditions on $\mathcal{A}$.

$\mathrm{H}(\mathcal{B}, \mathcal{C})$: The functions $\mathcal{B}\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $\mathcal{C}\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying the following structure conditions:

$$\begin{aligned} |\mathcal{B}(x,s,\xi)| &\le b_1|\xi|^{p\frac{q_1-1}{q_1}} + b_2|s|^{q_1-1} + b_3, && \text{for a.a. } x \in \Omega,\\ |\mathcal{C}(x,s)| &\le c_1|s|^{q_2-1} + c_2, && \text{for a.a. } x \in \partial\Omega, \end{aligned}$$

for all $s \in \mathbb{R}$, for all $\xi \in \mathbb{R}^N$, with positive constants b_j, c_k ($j \in \{1,2,3\}$, $k \in \{1,2\}$) and fixed numbers p, q_1, q_2 such that

$$1 < p < \infty, \qquad p \le q_1 \le p^*, \qquad p \le q_2 \le p_*.$$

Moreover, $\mathcal{C}$ satisfies the condition

$$|\mathcal{C}(x,s) - \mathcal{C}(y,t)| \le L\left[|x-y|^\alpha + |s-t|^\alpha\right], \quad |\mathcal{C}(x,s)| \le L$$

for all $(x,s),(y,t) \in \partial\Omega \times [-M_0, M_0]$ with $\alpha \in (0,1]$ and constants $M_0 > 0$ and $L \ge 0$.

Based on the hypotheses $\mathrm{H}(\mathcal{A})$ and $\mathrm{H}(\mathcal{B},\mathcal{C})$, problem (2.6.1) becomes

$$\begin{aligned} -\operatorname{div} \mathcal{A}(x, \nabla u) &= \mathcal{B}(x,u,\nabla u) && \text{in } \Omega,\\ \mathcal{A}(x,\nabla u) \cdot \nu &= \mathcal{C}(x,u) && \text{on } \partial\Omega. \end{aligned} \tag{2.6.41}$$

Combining Theorem 2.6.1 and the regularity theory of Lieberman [84] leads to the following result.

Theorem 2.6.2. *Let* $\Omega \subset \mathbb{R}^N$*,* $N > 1$*, be a bounded domain with a* $C^{1,\alpha}$*-boundary* $\partial\Omega$ *and let the assumptions* $H(\mathcal{A})$ *and* $H(\mathcal{B},\mathcal{C})$ *be satisfied. Then, every weak solution* $u \in W^{1,p}(\Omega)$ *of problem* (2.6.41) *belongs to* $C^{1,\beta}(\overline{\Omega})$ *for some* $\beta \in (0,1)$ *such that* $\beta = \beta(a_1, a_2, a_5, \alpha, N)$ *and*

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} \le C(a_1, a_2, a_3, a_5, N, \vartheta(1), M, \alpha, b_1, b_2, b_3)$$

where M *is the constant that comes from the statement of Theorem 2.6.1.*

Proof. We will apply Theorem 1.7 of Lieberman [84] and the comment after this theorem concerning global Hölder gradient estimates. First, we know from Theorem 2.6.1 that $\|u\|_\infty \le M$. The only thing we need to do is to check that the conditions (1.10a)–(1.10d) in [84, p. 320] are satisfied. From conditions $\mathrm{H}(\mathcal{A})$(iii), (ii) we see that the assumptions (1.10a) and (1.10b) are satisfied. Moreover, from H($\mathcal{B},\mathcal{C}$) and (2.6.38) we obtain

$$\begin{aligned}
|\mathcal{B}(x,s,\xi)| &\leq b_1|\xi|^{p\frac{q_1-1}{q_1}} + b_2|s|^{q_1-1} + b_3\\
&\leq b_1|\xi|^p + b_1 + b_2M^{q_1-1} + b_3\\
&= b_1|\xi|^{p-1}|\xi| + b_1 + b_2M^{q_1-1} + b_3\\
&\leq \frac{b_1}{a_3}\vartheta(|\xi|)|\xi| + b_1 + b_2M^{q_1-1} + b_3\\
&\leq \max\left\{\frac{b_1}{a_3}, b_1 + b_2M^{q_1-1} + b_3\right\}(\vartheta(|\xi|)|\xi| + 1)\,.
\end{aligned}$$

This proves condition (1.10d). Assumption (1.10c) follows from the fact that the function $\mathcal{A}$ is continuous differentiable in the space variable and independent of the s-variable. Then we may apply the mean value theorem which shows (1.10c). The desired result follows from Lieberman [84, Theorem 1.7] with the constants β, C as in that theorem (and their dependence on the data) and the constant M from Theorem 2.6.1. □

2.7 Further developments

1. We studied the boundedness of solutions to the boundary value problem (2.6.1) when the operators $\mathcal{A},\mathcal{B},\mathcal{C}$ satisfy the growth conditions (H). It would be interesting to know whether it is possible to obtain a similar result if we replace the constants a_i, b_j, c_k ($i \in \{1,\dots,6\}, j \in \{1,2,3\}, k \in \{1,2\}$) with functions in suitable L^s-spaces.

2. One could investigate if a result like Theorem 2.6.1 is still valid when we consider the whole space $\mathbb{R}^N$ instead of a bounded domain Ω. The main problem with the whole space is that some of the embeddings are no more valid.

3. Finally, it is possible to consider a system of equations instead of a single equation. More precisely, we are interested in studying the boundedness of solutions of the following problem

$$\begin{aligned}
-\operatorname{div}\mathcal{A}_1(x,u,\nabla u) &= \mathcal{B}_1(x,u,v,\nabla u,\nabla v) && \text{in } \Omega,\\
-\operatorname{div}\mathcal{A}_2(x,v,\nabla v) &= \mathcal{B}_2(x,u,v,\nabla u,\nabla v) && \text{in } \Omega,\\
\mathcal{A}_1(x,u,\nabla u)\cdot\nu_1 &= \mathcal{C}_1(x,u,v) && \text{on } \partial\Omega,\\
\mathcal{A}_2(x,v,\nabla v)\cdot\nu_2 &= \mathcal{C}_2(x,u,v) && \text{on } \partial\Omega.
\end{aligned} \tag{2.7.1}$$

where the operators $\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i$, $i = 1,2$, satisfy suitable structure conditions that can involve the critical exponents both in the domain and on the boundary.

For the sake of completeness, we incisively say that the boundedness of solutions of problem (2.7.1) is the object of a current collaboration with prof. P. Winkert.

Chapter 3

Singular systems in $\mathbb{R}^N$

3.1 Introduction

The increasing study of realistic mathematical models in population biology, whether we are dealing with a human population, bacterial viral growth or a population of an endangered species, is a reflection of their use in helping to understand the dynamic processes involved and in making practical predictions. Ecology, basically the study of the interrelationship between species and their environment in such areas as predator-prey, competition interactions or plant-herbivore systems, is now an enormous field. In order to give to the reader an idea of how these phenomena work, let us focus on the process of the growth of a living thing.

A fully developed organism is a complex arrangement of many different structures, yet it grows from a single fertilized cell. The formation of structures out a less structured tissue is known as *morphogenesis*. Though many processes are involved in morphogenesis, the result is a highly reproducible arrangement of the various structures. This variety arises because different cells develop in different ways, following a process known as *(biological) differentiation*. This is a complex procedure, which involves many phenomena like cell division, cell movement, gene activation and changes in the shape of the cells. Irrespective of the exact mechanism of differentiation, we point out that the way the cells develop depends on their position in the tissue. Hence, there must exist some mechanism to 'tell' the cells where they are in the tissue, so we can assume that differentiation is triggered by a patterned signal, called the morphogenetic field. Therefore, morphogenesis is a *(biological) pattern formation*.

This brief digression doesn't want to provide a full description of the differentiation process, instead focuses on one of the mechanisms that generate spatial information in the developing tissue. The choice of this mechanism was purely made on mathematical grounds, and points to the reaction-diffusion models, that have been used to reproduce biological patterns. The aim of this section is to present some deterministic models by way of an introduction to the field.

3.1.1 Continuous growth models

Single-species models are of relevance to laboratory studies in particular but, in the real world, can reflect a variety of effects which influence the population dynamics.

Let $N(t)$ be the population of the species at time t, then the rate of change

$$\frac{dN}{dt} = \text{births} - \text{ deaths} + \text{ migration} \tag{3.1.1}$$

is a conservation equation for the population. The form of the various terms on the right-hand side of (3.1.1) necessitates modeling the situation with which we are concerned. The simplest model has no migration and the birth and death terms are proportional to N. This means that

$$\frac{dN}{dt} = bN - dN \quad \Rightarrow \quad N(t) = N_0 e^{(b-d)t},$$

where b, d are positive constants, with initial population $N(0) = N_0$. Thus if $b > d$ the population grows exponentially while if $b < d$ it dies out. This approach, due to Malthus [89], is fairly unrealistic. Later, Verhulst [141, 142] proposed that a self-limiting process should operate when a population becomes too large. He suggested that

$$\frac{dN}{dt} = rN(1 - N/K), \tag{3.1.2}$$

where $r, K > 0$. This he called *logistic growth* in a population. In this model the per capita birth rate is $r(1 - N/K)$ and depends on N. The constant K is the *carrying capacity* of the environment, which is usually determined by the available sustaining resources.

There are two steady states for (3.1.2), namely, $N = 0$ and $N = K$, which occur when $dN/dt = 0$. The state $N = 0$ is unstable since linearization around it gives $dN/dt \simeq rN$, and so N grows exponentially from any small initial value. On the other hand, the equilibrium $N = K$ is stable: indeed, linearization around it gives $d(N - K)/dt \simeq -r(N - K)$ which means that $N \to K$ as $t \to +\infty$. The parameter K determines the size of the stable steady state population while r is a measure of the rate at which it is reached. The solution of (3.1.2) is

$$N(t) = \frac{N_0 K e^{rt}}{[K + N_0(e^{rt} - 1)]} \to K \quad \text{as } t \to +\infty. \tag{3.1.3}$$

The main feature of (3.1.3) is that it is particularly convenient to take when seeking qualitative dynamic behavior in populations in which $N = 0$ is an unstable steady state and $N(t)$ tends to a finite positive stable steady state. It occurs in a variety of different contexts because of its algebraic simplicity and also because it provides a preliminary qualitative idea of what happens with more realistic forms.

3.1.2 Models for interacting populations

When species interact, the population dynamics of each species is affected. In general there is a whole web of interacting species which moves towards structurally complex communities. Throughout this section, we will consider systems involving two or more species, giving special emphasis on two-species systems. There are three main types of interaction. If the growth rate of one population is decreasing while the other is increasing with respect to the first, then the populations are in a *predator-prey* situation. If the growth rate of each population is decreasing with respect to the other then we have *competition*. Finally, if each population's growth rate is enhancing then we have *mutualism* or *symbiosis*.

The Predator-Prey Model

Volterra first proposed a simple model for the predation of one species by another to explain the oscillatory levels of certain fish catches in the Adriatic (see [143]). If $N(t)$ is the prey population and $P(t)$ that of the predator at time t then Volterra's model is

$$\begin{aligned} \frac{dN}{dt} &= N(a - bP), \\ \frac{dP}{dt} &= P(cN - d), \end{aligned} \tag{3.1.4}$$

where $a, b, c, d > 0$. The assumptions in the model are the following: the prey in the absence of any predator grows unboundedly in a Malthusian way; the effect of the predation is to reduce the prey's per capita growth rate by a term proportional to both populations; in the absence of any prey for sustenance the predator's death rate decays exponentially; the prey's contribution to the predators' growth rate is proportional to the available prey as well as to the size of the predator population.

System (3.1.4) is known as the *Lotka-Volterra model* since the same equations were derived by Lotka from a chemical reaction which could exhibit periodic behavior in the chemical concentrations, see [88].

A phase plane analysis shows that the solutions of (3.1.4) are not structurally stable. This is a major inadequacy of such system, since any small perturbation can have a very marked effect, and this makes the system of little use for real interacting populations. One of the unrealistic assumptions in (3.1.4) is that the prey growth is unbounded in the absence of predation. To be more realistic these growth rates should depend on both the prey and the predator densities. Anyway this model, unrealistic though it is, does suggest that simple predator-prey interactions can result in periodic behavior of the populations. Reasoning heuristically this is not unexpected since if a prey population increases, it encourages growth of its predator. More predators however consume more prey, the population of which starts to decline. With less food around the predator population declines and when it is low enough, this allows the prey population to increase and the whole cycle starts over again.

The Competition Model

This model describes the situation in which two or more species inhibit each other's growth because, for example, they compete for territory or for the same limited food sources. In what follows we discuss a very simple competition model which demonstrates a fairly general principle observed in Nature: namely, when two species compete for the same limited resources, one of them usually becomes extinct. Consider the basic Lotka-Volterra model with each species N_1 and N_2 having logistic growth in the absence of the other. We thus have the following

$$\begin{aligned} \frac{dN_1}{dt} &= r_1 N_1 [1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_1}], \\ \frac{dN_2}{dt} &= r_2 N_2 [1 - \frac{N_2}{K_2} - b_{21}\frac{N_1}{K_2}], \end{aligned}$$

where $r_1, K_1, r_2, K_2, b_{12}, b_{21} > 0$, the r's are the linear birth rates while the K's are the carrying capacities. The coefficients b_{12}, b_{21} measure the competitive effect of

N_2 on N_1 and N_1 on N_2, respectively: they are in general not equal. As before, through the phase plane analysis we get the steady states, which have ecological implications. If there are two stable steady states, then both species can exist, and so they simply adjust to a lower population size than if there were no competition; in other words, the competition is not aggressive. If there are three nontrivial steady states, but only two of them are stable, then the survival will crucially depend on the advantage each species has. If the interspecific competition of one species is much stronger than the other, then one species dominates and the other species dies out. And finally, if the carrying capacities are sufficiently different, then one species becomes extinct.

The Mutualism or Symbiosis

There are many examples where the interaction of two or more species is to the advantage of all. Mutualism often plays a crucial role in promoting and even maintaining such species: plant and seed dispersal is one example. Even if survival is not at stake, the mutual advantage of symbiosis can be very important. As a topic of theoretical ecology, this area has not been as widely studied as the others, even though its importance is comparable to that of predator-prey and competition interactions. This is in part due to the fact that simple models in the Lotka-Volterra vein give silly results. The simplest mutualism model equivalent to the classical predator-prey one is

$$\frac{dN_1}{dt} = r_1 N_1 + a_1 N_1 N_2,$$
$$\frac{dN_2}{dt} = r_2 N_2 + a_2 N_2 N_1,$$

where $r_i, a_i > 0, i = 1, 2$. Since $dN_1/dt > 0$ and $dN_2/dt > 0$, then N_1 and N_2 simply grow unboundedly in mutual benefaction.

3.2 The reaction diffusion system

In an assemblage of particles (cells, bacteria, chemicals, animals and so on), each particle usually moves around in a random way. The particles spread out as a result of this irregular individual motion. When this microscopic irregular movement results in some macroscopic regular motion of the group we can think of it as a *diffusion* process. Let S be an arbitrary surface enclosing a volume V. The general conservation equation says that the rate of change of the amount of material in V is equal to the rate of flow of material across S into V plus the material created in V. Thus,

$$\frac{\partial}{\partial t} \int_V c(\mathbf{x}, t) dv = - \int_S \mathbf{J} \cdot \mathbf{ds} + \int_V f dv,$$

where $c(\mathbf{x}, t)$ is the concentration of the species while $\mathbf{J}$ and f are the flux and the source of material, respectively. Applying the divergence theorem to the surface integral and assuming $c(\mathbf{x}, t)$ is continuous, the last equation becomes

$$\int_V \left[\frac{\partial c}{\partial t} + \nabla \cdot \mathbf{J} - f(c, \mathbf{x}, t) \right] dv = 0.$$

Since the volume V is arbitrary, the integrand must be zero and so the *conservation equation* for c is the following

$$\frac{\partial c}{\partial t} + \nabla \cdot \mathbf{J} = f(c, \mathbf{x}, t). \tag{3.2.1}$$

Equation (3.2.1) holds for a general flux transport $\mathbf{J}$. Suppose that there are several interacting species. We then have a vector $u_i(\mathbf{x}, t)$, $i = 1, \dots, m$ of densities each diffusing with its own diffusion coefficient D_i and interacting according to the vector source term $\mathbf{f}$ which may depend on the densities themselves. If the flux has the particular form $\mathbf{J} = -D\nabla c$, then (3.2.1) becomes

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{f} + \nabla \cdot (D\nabla \mathbf{u}). \tag{3.2.2}$$

Equation (3.2.2) is referred to as a *reaction diffusion system.* Such a mechanism was proposed as a model for the chemical basis of morphogenesis by Turing [138], and started to be widely studied since about 1970.

Reaction-diffusion equations have found a considerable amount of interest in the last decades since they arise naturally in a variety of models from theoretical physics, chemistry and biology. The study of these phenomena needs a variety of different methods from many areas of mathematics: among others, numerical analysis, bifurcation and stability theory, semigroup theory, singular perturbations, phase space and topological methods.

Diffusion models form a reasonable basis for studying insect and animal dispersal and invasion. Dispersal of interacting and competing species was discussed in [121] and [122], respectively, while in [71] it has been shown that many species appear to disperse according to a reaction diffusion model with a constant diffusion coefficient. In what follows, we present the main mathematical features of these models.

3.2.1 The predator-prey model

The Lotka-Volterra system, whose ordinary version was already investigated in Section 3.1.2, reflects only population changes due to predation in a situation where predator and prey densities are not spatially dependent. However, it does not take into account either the fact that population is usually not homogeneously distributed, nor the fact that predators and preys naturally develop strategies for survival. Both of these considerations involve diffusion processes which can be quite intricate as different concentration levels of predators and preys may cause different population movements. Such movements can be determined by the concentration of the same species (diffusion) and that of other species (cross-diffusion). An example of such system is the following

$$\begin{aligned}
&u_t = \Delta[(d_1 + \rho_{12}v)u] + u(a_1 - b_1 u - c_1 v) && \text{in } \Omega \times (0, +\infty),\\
&v_t = \Delta[(d_2 + \rho_{21}u)v] + v(a_2 + b_2 u - c_2 v) && \text{in } \Omega \times (0, +\infty),\\
&u = v = 0 && \text{on } \partial\Omega \times (0, +\infty),
\end{aligned} \tag{3.2.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$. Moreover, $\rho_{12}, \rho_{21} \geq 0$ and $d_i, a_i, b_i, c_i, i = 1, 2$, are all positive constants except for a_2 which may be nonpositive. In particular, for u fixed, the right-hand side of the first equation of (3.2.3) is *increasing* in v and similarly, for fixed v, the right-hand side of the second equation of (3.2.3) is *increasing* in u. System (3.2.3) is known as the Lotka-Volterra predator-prey system with *cross-diffusion* effects. Here u and v, respectively, represent the population densities of prey and predator species which are interacting and migrating in the same habitat Ω. The coefficient d_i represents the natural dispersive force of movement of an individual, while ρ_{ij} describes the mutual interferences between individuals; ρ_{12} and ρ_{21} are usually referred to as cross-diffusion pressures. For example, ρ_{12} means the tendency that the prey keeps away from the predator. The boundary condition means that the habitat Ω is surrounded by a hostile environment.

When looking to steady-state solutions of (3.2.3) one is led, after a suitable rescaling, to the following

$$\begin{aligned} &\Delta[(1+\alpha v)u] + u(a-u-cv) = 0 && \text{in } \Omega, \\ &\Delta[(1+\beta u)v] + v(b+du-v) = 0 && \text{in } \Omega, \\ &u = v = 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.2.4}$$

Most research is concerned with the existence of positive solutions of (3.2.4). When there are no cross-diffusion effects ($\alpha = \beta = 0$), it is reduced to the classical predator-prey system, which has been extensively studied by many authors (see for instance [17, 87]). In particular, [87, Theorem 3.1] gives the exact range of parameter (a, b, c, d) for the existence of a positive solution of (3.2.4).

3.2.2 The competitive model

A central problem in population ecology is the understanding of the interactions between competing species. To this aim, different models based on reaction-diffusion equations have been developed with the aim to investigate phenomena of coexistence and exclusion for such species. A typical example of such a system is the following

$$\begin{aligned} &u_t(x,t) - d_1\Delta u = u(a_1 - b_1u - c_1v) = 0 && \text{in } \Omega\times(0,+\infty), \\ &v_t(x,t) - d_2\Delta v = v(a_2 - b_2v - c_2u) = 0 && \text{in } \Omega\times(0,+\infty), \\ &u = v = 0 && \text{on } \partial\Omega\times(0,+\infty), \end{aligned} \tag{3.2.5}$$

where Ω is an open bounded set of $\mathbb{R}^N$. This system models the situation where two species of densities u and v co-exist in Ω. The constants $d_1, d_2 > 0$ give the rates at which the species diffuse. The constants a_1, a_2 give, if positive, the net birth rates of the species and, if negative, their net death rates. The coefficients b_1, b_2 account for self-regulation of each species. In particular, if we assume $b_1, b_2 > 0$, this assures that the species are self-limiting, i.e., u and v must remain bounded as $t \to +\infty$. Finally, $c_1, c_2 > 0$, which means that, for u fixed, the right-hand side of the first equation of (3.2.5) is *decreasing* in v and, similarly, for fixed v, the right-hand side of the second equation of (3.2.5) is *decreasing* in u.

When looking for the steady-state solutions of (3.2.5) satisfying homogeneous Dirichlet boundary conditions, we are led to the following system

$$\begin{aligned} -d_1\Delta u &= u(a_1 - b_1u - c_1v) && \text{in } \Omega, \\ -d_2\Delta v &= v(a_2 - b_2v - c_2u) && \text{in } \Omega, \\ u &= v = 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.2.6}$$

Since, in the usual interpretation of the competition model, u and v are population variables, it is natural to consider only non-negative solutions of (3.2.6). There is a clearly *trivial* solution $u = v = 0$ for all values of the parameters. In addition, for some values of the parameters, there exist positive so-called *semi-trivial* solutions, where one of the dependent variables is identically zero and the other nonzero on Ω, that is $(u, v) = (u, 0)$ or $(0, v)$. If both u and v are positive for $x \in \Omega$, it is referred to as a *coexistence state*.

3.3 The slow diffusion

Many nonlinear problems in physics and mechanics, such as problems of fluid mechanics, reaction-diffusion problems, non-Newtonian fluid flows and fluid flows through certain types of porous media are formulated by partial differential equations that contains the p-Laplacian. For example, the equation of turbulent filtration in porous media, after a suitable rescaling leads to the following

$$\frac{\partial u}{\partial t} = \Delta_p u, \tag{3.3.1}$$

also known as the *evolution p-Laplacian equation*. Equation (3.3.1) is degenerate if $p > 2$ and singular if $1 < p < 2$. The case $p > 2$ is called the *slow diffusion* and the case $p < 2$ is the *fast diffusion*. The slow diffusion case is particularly well-suited to model diffusion processes related to population growths. Suppose a species is initially localized in space: then linear diffusion dictates *immediate* spreading of the population *everywhere in space*, whereas the nonlinear, slow-diffusion equation (3.3.1) predicts finite speed of propagation of the populated region. Clearly the second nonlinear model best describes real-life diffusion processes. Consider now the following equation

$$\frac{\partial u}{\partial t} = \Delta_p u + \lambda u^q,$$

where $p > 1, q > 0$ and $\lambda \in \mathbb{R}\backslash\{0\}$. Here the term λu^q describes the nonlinear source in the diffusion process, called 'heat source' if $\lambda > 0$ and 'cold source' if $\lambda < 0$. The appearance of nonlinear sources exerts a great influence to the properties of solutions and the influence of 'heat source' and 'cold source' is completely different.

Finally, consider a weakly coupled quasilinear elliptic system with homogeneous Dirichlet data

$$\begin{aligned} -\Delta_p u &= f(x, u, v) && \text{in } \Omega, \\ -\Delta_q v &= g(x, u, v) && \text{in } \Omega, \\ u = v &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.3.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and f, g are suitable functions. Systems of the above form are mathematical models occurring in studies of generalized reaction-diffusion theory, non-Newtonian fluid theory, non-Newtonian filtration and turbulent flow of a gas in a porous medium. In the non-Newtonian fluid theory, the quantity (p, q) is characteristic of the medium. Media with $(p, q) > (2, 2)$ are called *dilatant fluids* while if $(p, q) < (2, 2)$ they are *pseudo-plastic fluids*. If $(p, q) = (2, 2)$, they are *Newtonian fluids*. Moreover, systems like (3.3.2) describe nonlinear phenomena such as chemical reactions, pattern formation, population evolution where, for example, u and v represent the concentrations of two species in the process. In the framework described in the previous section, we are mainly interested in the slow diffusion case with suitable monotonicity assumptions on the reactions f and g in (3.3.2).

Several methods have been used to treat quasilinear equations and systems. In the scalar case, weak solutions can be obtained through variational methods which provide critical points of the corresponding energy functional. This approach is also fruitful in the case of potential systems, that is when the nonlinearities on the right hand side are the gradient of a C^1-functional. However, due to the loss of the variational structure, the treatment of nonvariational systems is more complicated and is based mostly on topological methods. Recently, there have been significant studies in this direction. Some existence and uniqueness result have been obtained with the assumption that (3.3.2) is the semilinear Lane-Emden system, which means $f = f(v)$ and $g = g(u)$ (see [35, 40]). The quasilinear Lane-Emden system was instead studied in [62]. When f and g are generic functions with suitable growth conditions some existence results, mainly based on the comparison principle and the Schauder's fixed point theorem, were obtained in [70]

3.4 Singular elliptic systems

In the previous section we described what happens from the physical and biological point of view when we deal with systems of elliptic regular equations. Here with the term regular we mean an equation which does not contain terms that could go to $+\infty$ as the variables approach zero. The aim of this section is to analyze the *singular* case, i.e., when the situation described above could happen. Singular boundary value problems for a single equation have been widely studied in the past several decades. While starting out as the study of ordinary differential equations in [127, 132], it rapidly progressed to the study of elliptic nonlinear boundary value problems, see [39, 78]. Presently, many of the earlier results have been generalized to different operators, often quasilinear and anisotropic, which often occur in applications such as fluid dynamics (see [28, 32, 33, 34]).

Although there is a substantial literature on systems of regular elliptic partial differential equations, to the best of our knowledge there has been no similar study of results for systems of singular elliptic problems even though they arise naturally in applications.

3.4.1 The Gierer-Meinhardt model

In 1972 Gierer and Meinhardt [57] proposed a mathematical model for pattern formation of spatial tissue structures in morphogenesis. The mechanism behind their model is based on the existence of two chemical substances: a slowly diffusing activator and a rapidly diffusing inhibitor. The ratio of their diffusion rates is assumed to be small.

The Gierer-Meinhardt model reads as

$$\begin{aligned}
&u_t = d_1\Delta u - \alpha u + c\rho\frac{u^p}{v^q} && \text{in } \Omega\times(0,T),\\
&v_t = d_2\Delta v - \beta v + c'\rho'\frac{u^r}{v^s} && \text{in } \Omega\times(0,T),\\
&\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 && \text{on } \partial\Omega\times(0,T),
\end{aligned} \tag{3.4.1}$$

in a smooth bounded domain Ω of $\mathbb{R}^N$, while n is the outer unit normal of Ω. Here the unknowns u, v stand for the concentration of activator and inhibitor with the source distribution ρ, ρ', respectively. Moreover, d_1, d_2 are the diffusion coefficients and α, β, c, $c' > 0$. Finally, the exponents $p, q, r, s > 0$ verify the relation $qr > (p-1)(s+1) > 0$. Using the original terminology of [57], we say that the activator and inhibitor sources are different when $q \neq s$.

Many works have been devoted to the study of the steady-state solutions of (3.4.1); that is, solutions of the stationary system

$$\begin{aligned} & d_1 \Delta u - \alpha u + c\rho \frac{u^p}{v^q} = 0 && \text{in } \Omega, \\ & d_2 \Delta v - \beta v + c'\rho' \frac{u^r}{v^s} = 0 && \text{in } \Omega, \\ & \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.4.2}$$

Problem (3.4.2) is quite difficult to solve in general, since it does not have a *variational structure*. A first step is to study its *shadow system*, an idea due to Keener [72]. Let us observe that dividing the second equation by d_2, letting formally $d_2 \to +\infty$ and making use of the boundary conditions we obtain that $v = \xi \equiv$ *constant* and the corresponding system is

$$\begin{aligned} & d_1 \Delta u - \alpha u + c\rho \frac{u^p}{\xi^q} = 0 && \text{in } \Omega, \\ & \xi^{s+1} = \frac{c\rho'}{\beta|\Omega|} \int_\Omega u^r && \\ & u > 0 && \text{in } \Omega, \\ & \frac{\partial u}{\partial n} = 0 && \text{on } \partial\Omega. \end{aligned}$$

New features of Gierer-Meinhardt type systems are emphasized in [54], where the author considered the following

$$\begin{aligned} & \Delta u - \alpha u + \frac{u^p}{v^q} + \rho(x) = 0, \;\; u > 0, && \text{in } \Omega, \\ & \Delta v - \beta v + \frac{u^r}{v^s} = 0, \;\; v > 0, && \text{in } \Omega, \\ & u = 0, \; v = 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.4.3}$$

in a smooth bounded domain Ω. Here $\rho \in C^{0,\gamma}(\overline{\Omega})$, $0 < \gamma < 1$, represents the source distribution of the activator. We assume that $\rho \geq 0$, with $\rho \not\equiv 0$, in Ω while $\alpha, \beta \geq 0$. Let us notice that the homogeneous Dirichlet boundary condition in (3.4.3), instead of the Neumann boundary condition as in (3.4.2), turns the system singular. Indeed, the nonlinearities u^p/v^q and u^r/v^s become unbounded around the boundary.

3.4.2 The quasilinear case

In this subsection we consider a quasilinear generalization of (3.4.2), namely

$$\begin{aligned} -\Delta_p u &= f(u, v) && \text{in } \Omega, \\ -\Delta_q v &= g(u, v) && \text{in } \Omega, \\ u, v &> 0 && \text{in } \Omega, \\ u, v &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.4.4}$$

on a bounded domain Ω with a $C^{1,\alpha}$ boundary $\partial\Omega$, $\alpha \in (0, 1)$, while $1 < p, q \leq N$.

Many important singular problems fit the setting of (3.4.4). For example, much interest was devoted to the case when

$$f(u,v) = u^{\alpha_1} v^{\beta_1} \quad \text{and} \quad g(u,v) = u^{\alpha_2} v^{\beta_2}, \tag{3.4.5}$$

with $\alpha_i, \beta_i \in \mathbb{R}$ being possibly negative. System (3.4.4) with the growth condition (3.4.5) was examined in [55, 56, 98] under the assumption that the associated system has a cooperative structure, that is, $\alpha_1, \beta_2 < 0 < \alpha_2, \beta_1$. In all these papers the authors obtained existence results via different techniques: an iterative scheme constructed through a sub-supersolution [55], a fixed point argument in a sub-supersolution setting [56], a sub-supersolution method for system combined with perturbation techniques [98].

Another important class of singular problems incorporated in statement (3.4.4) patterns the system for

$$f(u,v) = u^{\alpha_1} + v^{\beta_1} \quad \text{and} \quad g(u,v) = u^{\alpha_2} + v^{\beta_2},$$

with possibly negative exponents. Relevant contributions on this topic can be found in [3, 46]. In [99] the authors instead considered a more general class of functions f, g, satisfying the following hypotheses:

$$\begin{aligned} & f(s_1, s_2) \leq m_1(1 + s_1^{\alpha_1})(1 + s_2^{\beta_1}) \quad \forall\, s_1, s_2 > 0, \\ & \text{with } m_1 > 0 \text{ and } \alpha_1, \beta_1 < 0 \text{ such that } \alpha_1 + \beta_1 > -1, \end{aligned} \tag{3.4.6}$$

and

$$\begin{aligned} & g(s_1, s_2) \leq m_2(1 + s_1^{\alpha_2})(1 + s_2^{\beta_2}) \quad \forall\, s_1, s_2 > 0, \\ & \text{with } m_2 > 0 \text{ and } \alpha_2, \beta_2 < 0 \text{ such that } \alpha_2 + \beta_2 > -1. \end{aligned} \tag{3.4.7}$$

The main technical difficulty consists in the presence of singular terms that can occur under hypotheses (3.4.6)-(3.4.7). Indeed, the imposed hypotheses do not guarantee that the Euler functional associated to problem (3.4.4) is well defined, so the variational method can not be applied. In addition, the method of sub-supersolution in its system version does not work for problem (3.4.4) due to its noncooperative character, which means that generally the functions $f(u, \cdot\,)$ and $g(\,\cdot, v)$ are not necessarily increasing whenever u, v are fixed. To handle this problem, the authors in [99] exploited the behavior toward zero and infinity of the nonlinearities f, g by introducing adequate truncations. This gave rise to a regularized system for (3.4.4) depending on a parameter $\varepsilon > 0$ whose study is relevant for the initial problem. By applying Schauder's fixed point theorem, they showed that the regularized system has a positive and sufficiently regular solution $(u_\varepsilon, v_\varepsilon)$. Moreover, this solution is located in some rectangle which does not contain zero for all $\varepsilon > 0$, whose lower bound is independent of ε whereas the upper bound can depend on ε. Then, a positive solution of (3.4.4) is obtained by passing to the limit as $\varepsilon \to 0$. Their arguments are based on a-priori estimates, Hardy-Sobolev inequality and the (S_+)-property of the negative p-Laplacian.

In [100] different classes of functions were considered, namely those satisfying the following

$$\begin{aligned} & m_1 s_1^{\alpha_1} s_2^{\beta_1} \leq f(s_1, s_2) \leq M_1 s_1^{\alpha_1} s_2^{\beta_1} \quad \forall\, s_1, s_2 > 0, \text{ with } M_1, m_1 > 0 \\ & \text{and } \alpha_1 \in \mathbb{R}, \beta_1 < 0 \text{ such that } |\alpha_1| - \beta_1 < \min(1, p-1), \end{aligned}$$

and

$$m_2 s_1^{\alpha_2} s_2^{\beta_2} \leq g(s_1, s_2) \leq M_2 s_1^{\alpha_2} s_2^{\beta_2} \quad \forall\, s_1, s_2 > 0, \text{ with } M_2, m_2 > 0$$
$$\text{and } \beta_2 \in \mathbb{R}, \alpha_2 < 0 \text{ such that } |\beta_2| - \alpha_2 < \min(1, q-1),$$

A basic feature of this setting is that now the singularity in problem (3.4.4) comes out through a competitive structure of the nonlinearities $f(u, v)$ and $g(u, v)$. It is caused by the fact that the parameters β_1, α_2 are negative, which prevents f and g to be increasing with respect to v and u, respectively. Due to this, the sub-supersolution method is not directly applicable without additional assumptions. In order to establish the existence of positive solutions, the authors developed some comparison arguments, which allowed them to get an auxiliary result that provides a-priori estimates. In turn, these estimates enabled them to obtain the result again by applying the Schauder's fixed point theorem to a fixed point problem associated to (3.4.4).

3.4.3 Singular elliptic systems in the whole space $\mathbb{R}^N$

In Section 3.4.1 we saw how Gierer and Meinhardt proposed system (3.4.1) as a model of biological pattern formation, making particular attention to its stationary version (3.4.2). Let us consider the following particular case of (3.4.2)

$$\begin{aligned} &d\Delta a - a + a^2/h = 0 && \text{in } \Omega, \\ &D\Delta h - h + a^2 = 0 && \text{in } \Omega, \\ &\frac{\partial a}{\partial h} = \frac{\partial v}{\partial n} = 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.4.8}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N = 1, 2$. The associated shadow system reads as

$$\begin{aligned} &d\Delta a - a + a^2/\xi = 0 && \text{in } \Omega, \\ &\xi = \frac{1}{|\Omega|}\int_\Omega a^2 \\ &a > 0 && \text{in } \Omega, \\ &\frac{\partial a}{\partial n} = 0 && \text{on } \partial\Omega. \end{aligned}$$

After the rescaling $w(y) = \xi^{-1} a(d^{1/2} y)$ we obtain the following scalar equation

$$\begin{aligned} &\Delta w - w + w^2 = 0 && \text{in } \Omega_d, \\ &w > 0 && \text{in } \Omega_d, \\ &\frac{\partial w}{\partial n} = 0 && \text{on } \partial\Omega_d, \end{aligned} \tag{3.4.9}$$

where Ω_d denotes the expanding domain $d^{-1/2}\Omega$. The study of nonconstant solutions of (3.4.9) and the related equations as d approaches zero has been the object of extensive research in recent years. Since the domain is expanding as $d \to 0$, it is natural to search for solutions w which resemble, after a convenient translation of the origin, a solution of the limiting problem

$$\begin{aligned} &\Delta w - w + w^2 = 0 && \text{in } \mathbb{R}^N, \\ &0 < w(y) \to 0 && \text{as } |y| \to \infty. \end{aligned}$$

It is well known that this problem has solutions for $N \leq 5$, which are unique up to translations and radially symmetric. It is of course natural to ask whether these solutions will actually correspond to limiting configuration solutions of the full system when D becomes finite and d very small. Let us make in (3.4.8) the scaling

$$u(x) = \sigma^{-1} a(d^{1/2}x), \quad v(x) = \sigma^{-1} h(d^{1/2}x).$$

Then, letting $d \to 0$ we obtain, for $N = 2$, the *limiting system*

$$\begin{aligned} &\Delta u - u + u^2/v = 0 && \text{in } \mathbb{R}^2, \\ &\Delta v - \sigma^2 v + u^2 = 0 && \text{in } \mathbb{R}^2, \\ &u, v > 0, \quad u, v \to 0 && \text{as } |x| \to +\infty. \end{aligned}$$

This setting is rather natural, since it may correspond to a very large domain with the pattern formation process taking place only very far away from the boundary.

Of course problem (3.4.8) can be generalized to the case $N \geq 3$. This was made for example in [101], where the authors considered the following system

$$\begin{aligned} &-\Delta u + \alpha(x)u = h_1(x)\frac{1}{v^q} && \text{in } \mathbb{R}^N, \\ &-\Delta v + \beta(x)v = h_2(x)\frac{u^r}{v^s} && \text{in } \mathbb{R}^N, \\ &u, v > 0 && \text{in } \mathbb{R}^N, \\ &u(x), v(x) \to 0 && \text{as } |x| \to \infty. \end{aligned} \qquad (3.4.10)$$

Here α, β, h_1, h_2 are given, not necessarily continuous functions, $s \in]0,1[$ and $q, r > 0$ such that $r - s \leq 1$. The main idea for solving system (3.4.10) is to regularize it by introducing a suitable parameter ε, as made in Section 3.4.2, and then to solve the regularized system by applying the Schauder's fixed point theorem.

The main difficulty when dealing with problems in the whole space is the lack of compactness due to the translation and dilation invariance of $\mathbb{R}^N$. Moreover, some powerful tools as the Hardy-Sobolev inequality are no more at hand. To prove the existence of solution of problem (3.4.10) the authors applied the reasoning used in [4] to demonstrate the uniqueness of the solution to the following problem

$$\begin{aligned} &-\Delta u + a(x)u = h(x)u^{-\gamma} && \text{in } \mathbb{R}^N, \\ &u > 0 && \text{in } \mathbb{R}^N, \end{aligned}$$

where γ is a positive number.

System (3.4.8) could be also generalized to its quasilinear version. To the best of our knowledge, this problem was not yet studied in the literature, motivated by the discussion in Section 3.3. An attempt in this direction was made in our work [91], which is the subject of the next section.

3.5 Our results

We consider the following system of quasilinear elliptic equations

$$\begin{aligned} &-\Delta_{p_1} u = a_1(x) f(u, v) && \text{in } \mathbb{R}^N, \\ &-\Delta_{p_2} v = a_2(x) g(u, v) && \text{in } \mathbb{R}^N, \\ &u, v > 0 && \text{in } \mathbb{R}^N, \\ &u, v \to 0, && \text{as } |x| \to +\infty, \end{aligned} \qquad (3.5.1)$$

where $N \geq 3$ and $1 < p_i < N$. Nonlinearities $f, g\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ are continuous and fulfill the condition

($\mathrm{H}_{f,g}$) There exist $m_i, M_i > 0$, $i = 1, 2$, such that

$$m_1 s^{\alpha_1} \leq f(s,t) \leq M_1 s^{\alpha_1}(1 + t^{\beta_1}),$$
$$m_2 t^{\beta_2} \leq g(s,t) \leq M_2(1 + s^{\alpha_2})t^{\beta_2}$$

for all $s, t \in \mathbb{R}^+$, with $-1 < \alpha_1, \beta_2 < 0 < \alpha_2, \beta_1$,

$$\alpha_1 + \alpha_2 < p_1 - 1, \quad \beta_1 + \beta_2 < p_2 - 1, \tag{3.5.2}$$

as well as

$$\beta_1 < \frac{p_2^*}{p_1^*} \min\{p_1 - 1, p_1^* - p_1\}, \quad \alpha_2 < \frac{p_1^*}{p_2^*} \min\{p_2 - 1, p_2^* - p_2\}.$$

As usual, p_i^* denotes the critical Sobolev exponent corresponding to p_i, see Section 2.6.1. Coefficients $a_i : \mathbb{R}^N \to \mathbb{R}$ satisfy the assumption

(H_a) $a_i(x) > 0$ a.e. in $\mathbb{R}^N$ and $a_i \in L^1(\mathbb{R}^N) \cap L^{\zeta_i}(\mathbb{R}^N)$, where

$$\frac{1}{\zeta_1} \leq 1 - \frac{p_1}{p_1^*} - \frac{\beta_1}{p_2^*}, \quad \frac{1}{\zeta_2} \leq 1 - \frac{p_2}{p_2^*} - \frac{\alpha_2}{p_1^*}.$$

Let $\mathcal{D}^{1,p_i}(\mathbb{R}^N)$ be the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|w\|_{\mathcal{D}^{1,p_i}(\mathbb{R}^N)} := \|\nabla w\|_{L^{p_i}(\mathbb{R}^N)}.$$

Recall that

$$\mathcal{D}^{1,p_i}(\mathbb{R}^N) = \{w \in L^{p_i^*}(\mathbb{R}^N) : |\nabla w| \in L^{p_i}(\mathbb{R}^N)\};$$

see, e.g., [83, Theorem 8.3]. A pair $(u, v) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$ is called a weak solution to (3.5.1) provided $u, v > 0$ a.e. in $\mathbb{R}^N$, $u(x), v(x) \to 0$ as $|x| \to +\infty$ and

$$\int_{\mathbb{R}^N} |\nabla u|^{p_1-2} \nabla u \nabla \varphi \, dx = \int_{\mathbb{R}^N} a_1 f(u,v) \varphi \, dx,$$
$$\int_{\mathbb{R}^N} |\nabla v|^{p_2-2} \nabla v \nabla \psi \, dx = \int_{\mathbb{R}^N} a_2 g(u,v) \psi \, dx$$

for all $(\varphi, \psi) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$.

The most interesting aspect lies in the fact that both f and g can exhibit singularities through $\mathbb{R}^N$ which, without loss of generality, are located at zero. Indeed, $-1 < \alpha_1, \beta_2 < 0$ by ($\mathrm{H}_{f,g}$). This in general represents a serious difficulty to overcome.

We have already seen that singular systems in the whole space have been investigated only for $p := q := 2$, essentially exploiting the linearity of involved differential operators, see Section 3.4.3. Nevertheless, even in the semilinear case, (3.5.1) cannot be reduced to Gierer-Meinhardt's case once ($\mathrm{H}_{f,g}$) is assumed. Moreover, variational methods do not work, at least not in a direct way, because the Euler functional associated with problem (3.5.1) is not well defined. A similar comment holds for sub-supersolution techniques, that are usually employed in the case of bounded domains, as already seen. Hence, we were naturally led to apply fixed point results. An a-priori estimate in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ for weak solutions of (3.5.1) is first established (cf. Theorem 3.5.1) by a Moser's type iteration procedure and an adequate truncation which, due to singular terms, require a specific treatment. We next perturb (3.5.1) by introducing a parameter $\varepsilon > 0$. This produces the family of regularized systems

$$\begin{aligned} -\Delta_{p_1} u &= a_1(x) f(u+\varepsilon, v) \qquad && \text{in } \mathbb{R}^N, \\ -\Delta_{p_2} v &= a_2(x) g(u, v+\varepsilon) && \text{in } \mathbb{R}^N, \\ u, v &> 0 && \text{in } \mathbb{R}^N, \end{aligned} \tag{3.5.3}$$

whose study yields useful informations on (3.5.1). Indeed, the previous L^∞- boundedness still holds for solutions to (3.5.3), regardless of ε. Thus, via Schauder's fixed point theorem we get a weak solution $(u_\varepsilon, v_\varepsilon)$ lying inside a rectangle given by positive lower bounds, where ε does not appear, and positive upper bounds, that may instead depend on ε. Finally, letting $\varepsilon \to 0^+$ and using the $(\mathrm{S})_+$-property of the negative p-Laplacian in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ (see Lemma 3.5.3) yields a weak solution to (3.5.1).

3.5.1 Preliminaries

Let $\Omega \subseteq \mathbb{R}^N$ be a measurable set, $t \in \mathbb{R}$ and $w, z \in L^p(\mathbb{R}^N)$. As seen in Chapter 2, we write $|\Omega|$ for the Lebesgue measure of Ω, while we set

$$\Omega(w \le t) := \{x \in \Omega : w(x) \le t\} \quad \text{and} \quad \|w\|_p := \|w\|_{L^p(\mathbb{R}^N)}.$$

The meaning of $\Omega(w > t)$, etc. is analogous. By definition, $w \le z$ if $w(x) \le z(x)$ a.e. in $\mathbb{R}^N$.

Given $1 \le q < p$, neither $L^p(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ nor the reverse embedding hold true. However, the situation looks better for functions belonging to $L^1(\mathbb{R}^N)$, as the next proposition shows.

Proposition 3.5.1. *Suppose $p > 1$ and $w \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. Then, $w \in L^q(\mathbb{R}^N)$ whatever $q \in\,]1, p[$.*

Proof. Thanks to Hölder's inequality, with exponents p/q and $p/(p-q)$, and Chebyshev's inequality we have

$$\begin{aligned} \|w\|_q^q &= \int_{\mathbb{R}^N(|w|\le 1)} |w|^q dx + \int_{\mathbb{R}^N(|w|>1)} |w|^q dx \\ &\le \int_{\mathbb{R}^N(|w|\le 1)} |w|\, dx + \left(\int_{\mathbb{R}^N(|w|>1)} |w|^p dx \right)^{q/p} |\mathbb{R}^N(|w|>1)|^{1-q/p} \\ &\le \int_{\mathbb{R}^N} |w|\, dx + \left(\int_{\mathbb{R}^N} |w|^p dx \right)^{q/p} \left(\int_{\mathbb{R}^N} |w|^p dx \right)^{1-q/p} \\ &= \|w\|_1 + \|w\|_p^p. \end{aligned}$$

This completes the proof. ☐

The summability properties of a_i collected below will be exploited throughout the section.

Remark 3.5.1. *Let assumption (H_a) be fulfilled. Then, for any $i = 1, 2$, the following holds* (j$_1$) $a_i \in L^{(p_i^*)'}(\mathbb{R}^N)$;

(j$_2$) $a_i \in L^{\gamma_i}(\mathbb{R}^N)$, *where* $\gamma_i := 1/(1-t_i)$, *with*

$$t_1 := \frac{\alpha_1+1}{p_1^*} + \frac{\beta_1}{p_2^*}, \quad t_2 := \frac{\alpha_2}{p_1^*} + \frac{\beta_2+1}{p_2^*};$$

(j$_3$) $a_i \in L^{\delta_i}(\mathbb{R}^N)$, *for* $\delta_i := 1/(1-s_i)$ *and*

$$s_1 := \frac{\alpha_1+1}{p_1^*}, \quad s_2 := \frac{\beta_2+1}{p_2^*};$$

(j$_4$) $a_i \in L^{\xi_i}(\mathbb{R}^N)$, *where* $\xi_i \in]p_i^*/(p_i^*-p_i), \zeta_i[$.

To verify (j$_1$)–(j$_4$) *we simply note that* $\zeta_i > \max\{(p_i^*)', \gamma_i, \delta_i, \xi_i\}$ *and apply Proposition 3.5.1.*

Let us next show that the operator $-\Delta_p$ is of type $(\mathrm{S})_+$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$.

Proposition 3.5.2. *If* $1 < p < N$ *and* $(u_n) \subseteq \mathcal{D}^{1,p}(\mathbb{R}^N)$ *satisfies*

$$u_n \rightharpoonup u \text{ in } \mathcal{D}^{1,p}(\mathbb{R}^N), \tag{3.5.4}$$

$$\limsup_{n\to\infty} \langle -\Delta_p u_n, u_n - u\rangle \le 0, \tag{3.5.5}$$

then $u_n \to u$ *in* $\mathcal{D}^{1,p}(\mathbb{R}^N)$.

Proof. By monotonicity one has

$$\langle -\Delta_p u_n - (-\Delta_p u), u_n - u\rangle \ge 0 \quad \forall n \in \mathbb{N},$$

which evidently entails

$$\liminf_{n\to\infty} \langle -\Delta_p u_n - (-\Delta_p u), u_n - u\rangle \ge 0.$$

Via (3.5.4)–(3.5.5) we then get

$$\limsup_{n\to\infty} \langle -\Delta_p u_n - (-\Delta_p u), u_n - u\rangle \le 0.$$

Therefore,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u)\, dx = 0. \tag{3.5.6}$$

Since [108, Lemma A.0.5] yields

$$\begin{aligned}&\int_{\mathbb{R}^N} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u)\, dx \\ &\ge \begin{cases} C_p \int_{\mathbb{R}^N} \frac{|\nabla(u_n-u)|^2}{(|\nabla u_n|+|\nabla u|)^{2-p}}\, dx & \text{if } 1<p<2, \\ C_p \int_{\mathbb{R}^N} |\nabla(u_n-u)|^p\, dx & \text{otherwise} \end{cases} \qquad \forall n \in \mathbb{N},\end{aligned}$$

the desired conclusion, namely

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla(u_n-u)|^p\, dx = 0,$$

directly follows from (3.5.6) once $p \geq 2$. If $1 < p < 2$ then Hölder's inequality and (3.5.4) lead to

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla(u_n - u)|^p \, dx \\
&= \int_{\mathbb{R}^N} \frac{|\nabla(u_n - u)|^p}{(|\nabla u_n| + |\nabla u|)^{\frac{p(2-p)}{2}}} \left(|\nabla u_n| + |\nabla u|\right)^{\frac{p(2-p)}{2}} dx \\
&\leq \left(\int_{\mathbb{R}^N} \frac{|\nabla(u_n - u)|^2}{(|\nabla u_n| + |\nabla u|)^{2-p}} \, dx \right)^{\frac{p}{2}} \left(\int_{\mathbb{R}^N} (|\nabla u_n| + |\nabla u|)^p dx \right)^{\frac{2-p}{2}} \\
&\leq C \left(\int_{\mathbb{R}^N} \frac{|\nabla(u_n - u)|^2}{(|\nabla u_n| + |\nabla u|)^{2-p}} \, dx \right)^{\frac{p}{2}}, \quad n \in \mathbb{N},
\end{aligned}$$

with appropriate $C > 0$. Now, the argument goes on as before. □

3.5.2 Boundedness of solutions

The main result of this subsection, Theorem 3.5.1 below, provides an $L^\infty(\mathbb{R}^N)$-a priori estimate for weak solutions to (3.5.1). Its proof will be performed into three steps.

Lemma 3.5.1 ($L^{p_i^*}(\mathbb{R}^N)$-uniform boundedness)**.** *There exists $\rho > 0$ such that*

$$\max\left\{ \|u\|_{p_1^*}, \, \|v\|_{p_2^*} \right\} \leq \rho \tag{3.5.7}$$

for every $(u, v) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$ solving problem (3.5.1)*.*

Proof. Multiply both equations in (3.5.1) by u and v, respectively, integrate over $\mathbb{R}^N$, and use $(\mathrm{H}_{f,g})$ to arrive at

$$\begin{aligned}
\|\nabla u\|_{p_1}^{p_1} &= \int_{\mathbb{R}^N} a_1 f(u,v) u \, dx \leq M_1 \int_{\mathbb{R}^N} a_1 u^{\alpha_1+1}(1 + v^{\beta_1}) \, dx, \\
\|\nabla v\|_{p_2}^{p_2} &= \int_{\mathbb{R}^N} a_2 g(u,v) v \, dx \leq M_2 \int_{\mathbb{R}^N} a_2 (1 + u^{\alpha_2}) v^{\beta_2+1} \, dx.
\end{aligned}$$

Through the embedding $\mathcal{D}^{1,p_i}(\mathbb{R}^N) \hookrightarrow L^{p_i^*}(\mathbb{R}^N)$, besides Hölder's inequality, we obtain

$$\begin{aligned}
\|\nabla u\|_{p_1}^{p_1} &\leq M_1 \left(\|a_1\|_{\delta_1} \|u\|_{p_1^*}^{\alpha_1+1} + \|a_1\|_{\gamma_1} \|u\|_{p_1^*}^{\alpha_1+1} \|v\|_{p_2^*}^{\beta_1} \right) \\
&\leq C_1 \|\nabla u\|_{p_1}^{\alpha_1+1} \left(\|a_1\|_{\delta_1} + \|a_1\|_{\gamma_1} \|\nabla v\|_{p_2}^{\beta_1} \right);
\end{aligned}$$

cf. also Remark 3.5.1. Likewise,

$$\|\nabla v\|_{p_2}^{p_2} \leq C_2 \|\nabla v\|_{p_2}^{\beta_2+1} \left(\|a_2\|_{\delta_2} + \|a_2\|_{\gamma_2} \|\nabla u\|_{p_1}^{\alpha_2} \right).$$

Thus, a fortiori,

$$\begin{aligned}
\|\nabla u\|_{p_1}^{p_1-1-\alpha_1} &\leq C_1 \left(\|a_1\|_{\delta_1} + \|a_1\|_{\gamma_1} \|\nabla v\|_{p_2}^{\beta_1} \right), \\
\|\nabla v\|_{p_2}^{p_2-1-\beta_2} &\leq C_2 \left(\|a_2\|_{\delta_2} + \|a_2\|_{\gamma_2} \|\nabla u\|_{p_1}^{\alpha_2} \right),
\end{aligned} \tag{3.5.8}$$

which imply

$$\begin{aligned}
&\|\nabla u\|_{p_1}^{p_1-1-\alpha_1} + \|\nabla v\|_{p_2}^{p_2-1-\beta_2} \\
&\leq C_1 \left(\|a_1\|_{\delta_1} + \|a_1\|_{\gamma_1} \|\nabla v\|_{p_2}^{\beta_1} \right) + C_2 \left(\|a_2\|_{\delta_2} + \|a_2\|_{\gamma_2} \|\nabla u\|_{p_1}^{\alpha_2} \right).
\end{aligned}$$

Rewriting this inequality as

$$\begin{aligned}\|\nabla u\|_{p_1}^{\alpha_2}&\left(\|\nabla u\|_{p_1}^{p_1-1-\alpha_1-\alpha_2}-C_2\|a_2\|_{\gamma_2}\right)\\&+\|\nabla v\|_{p_2}^{\beta_1}\left(\|\nabla v\|_{p_2}^{p_2-1-\beta_1-\beta_2}-C_1\|a_1\|_{\gamma_1}\right)\\&\le C_1\|a_1\|_{\delta_1}+C_2\|a_2\|_{\delta_2},\end{aligned}\tag{3.5.9}$$

four situations may occur. If

$$\|\nabla u\|_{p_1}^{p_1-1-\alpha_1-\alpha_2}\le C_2\|a_2\|_{\gamma_2}\,,\quad \|\nabla v\|_{p_2}^{p_2-1-\beta_1-\beta_2}\le C_1\|a_1\|_{\gamma_1}$$

then (3.5.7) follows from (j_2) of Remark 3.5.1, conditions (3.5.2), and the embedding $\mathcal{D}^{1,p_i}(\mathbb{R}^N)\hookrightarrow L^{p_i^*}(\mathbb{R}^N)$. Assume next that

$$\|\nabla u\|_{p_1}^{p_1-1-\alpha_1-\alpha_2}> C_2\|a_2\|_{\gamma_2}\,,\quad \|\nabla v\|_{p_2}^{p_2-1-\beta_1-\beta_2}> C_1\|a_1\|_{\gamma_1}\,.\tag{3.5.10}$$

Thanks to (3.5.9) we have

$$\|\nabla u\|_{p_1}^{\alpha_2}(\|\nabla u\|_{p_1}^{p_1-1-\alpha_1-\alpha_2}-C_2\|a_2\|_{\gamma_2})\le C_1\|a_1\|_{\delta_1}+C_2\|a_2\|_{\delta_2},$$

whence, on account of (3.5.10),

$$\begin{aligned}\|\nabla u\|_{p_1}^{p_1-1-\alpha_1-\alpha_2}&\le\frac{C_1\|a_1\|_{\delta_1}+C_2\|a_2\|_{\delta_2}}{\|\nabla u\|_{p_1}^{\alpha_2}}+C_2\|a_2\|_{\gamma_2}\\&\le\frac{C_1\|a_1\|_{\delta_1}+C_2\|a_2\|_{\delta_2}}{\|a_2\|_{\gamma_2}^{\frac{\alpha_2}{p_1-1-\alpha_1-\alpha_2}}}+C_2\|a_2\|_{\gamma_2}.\end{aligned}$$

A similar inequality holds true for v. So, (3.5.7) is achieved reasoning as before. Finally, if

$$\|\nabla u\|_{p_1}^{p_1-1-\alpha_1-\alpha_2}\le C_2\|a_2\|_{\gamma_2}\,,\quad \|\nabla v\|_{p_2}^{p_2-1-\beta_1-\beta_2}> C_1\|a_1\|_{\gamma_1}\tag{3.5.11}$$

then (3.5.8) and (3.5.11) entail

$$\|\nabla v\|_{p_2}^{p_2-1-\beta_2}\le C_2\left[\|a_2\|_{\delta_2}+\|a_2\|_{\gamma_2}\left(C_2\|a_2\|_{\gamma_2}\right)^{\frac{\alpha_2}{p_1-1-\alpha_1-\alpha_2}}\right].$$

By (3.5.2) again we thus get

$$\max\{\|\nabla u\|_{p_1},\|\nabla v\|_{p_2}\}\le C_3\,,$$

where $C_3>0$. This yields (3.5.7), because $\mathcal{D}^{1,p_i}(\mathbb{R}^N)\hookrightarrow L^{p_i^*}(\mathbb{R}^N)$. The last case, i.e.,

$$\|\nabla u\|_{p_1}^{p_1-1-\alpha_1-\alpha_2}> C_2\|a_2\|_{\gamma_2}\,,\quad \|\nabla v\|_{p_2}^{p_2-1-\beta_1-\beta_2}\le C_1\|a_1\|_{\gamma_1}$$

is analogous. □

To shorten notation, we write

$$\mathcal{D}^{1,p_i}(\mathbb{R}^N)_+:=\{w\in\mathcal{D}^{1,p_i}(\mathbb{R}^N): w\ge 0\ \text{ a.e. in }\ \mathbb{R}^N\}.$$

Lemma 3.5.2 (Truncation)**.** *Let $(u,v)\in\mathcal{D}^{1,p_1}(\mathbb{R}^N)\times\mathcal{D}^{1,p_2}(\mathbb{R}^N)$ be a weak solution of* (3.5.1)*. Then,*

$$\int_{\mathbb{R}^N(u>1)}|\nabla u|^{p_1-2}\nabla u\nabla\varphi\,dx\le M_1\int_{\mathbb{R}^N(u>1)}a_1(1+v^{\beta_1})\varphi\,dx,\tag{3.5.12}$$

$$\int_{\mathbb{R}^N(v>1)}|\nabla v|^{p_2-2}\nabla v\nabla\psi\,dx\le M_2\int_{\mathbb{R}^N(v>1)}a_2(1+u^{\alpha_2})\psi\,dx\tag{3.5.13}$$

for all $(\varphi,\psi)\in\mathcal{D}^{1,p_1}(\mathbb{R}^N)_+\times\mathcal{D}^{1,p_2}(\mathbb{R}^N)_+$.

Proof. Pick a C^1 cut-off function $\eta\colon \mathbb{R}\to[0,1]$ such that

$$\eta(t)=\begin{cases}0 & \text{if } t\le 0,\\ 1 & \text{if } t\ge 1,\end{cases}\qquad \eta'(t)\ge 0\quad \forall\, t\in[0,1],$$

and, given $\delta>0$, define $\eta_\delta(t):=\eta\left(\frac{t-1}{\delta}\right)$. If $w\in\mathcal{D}^{1,p_i}(\mathbb{R}^N)$, then

$$\eta_\delta\circ w\in\mathcal{D}^{1,p_i}(\mathbb{R}^N),\quad \nabla(\eta_\delta\circ w)=(\eta_\delta'\circ w)\nabla w, \tag{3.5.14}$$

as a standard verification shows.

Fix now $(\varphi,\psi)\in\mathcal{D}^{1,p_1}(\mathbb{R}^N)_+\times\mathcal{D}^{1,p_2}(\mathbb{R}^N)_+$. Multiply the first equation in (3.5.1) by $(\eta_\delta\circ u)\varphi$, integrate over $\mathbb{R}^N$, and use $(\mathrm{H}_{f,g})$ to achieve

$$\int_{\mathbb{R}^N}|\nabla u|^{p_1-2}\nabla u\nabla((\eta_\delta\circ u)\varphi)\,dx\le M_1\int_{\mathbb{R}^N}a_1u^{\alpha_1}(1+v^{\beta_1})(\eta_\delta\circ u)\varphi\,dx.$$

By (3.5.14) we have

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla u|^{p_1-2}\nabla u\nabla((\eta_\delta\circ u)\varphi)\,dx\\ &\quad=\int_{\mathbb{R}^N}|\nabla u|^{p_1}(\eta_\delta'\circ u)\varphi\,dx+\int_{\mathbb{R}^N}(\eta_\delta\circ u)|\nabla u|^{p_1-2}\nabla u\nabla\varphi\,dx,\end{aligned}$$

while $\eta_\delta'\circ u\ge 0$ in $\mathbb{R}^N$. Therefore,

$$\int_{\mathbb{R}^N}(\eta_\delta\circ u)|\nabla u|^{p_1-2}\nabla u\nabla\varphi\,dx\le M_1\int_{\mathbb{R}^N}a_1u^{\alpha_1}(1+v^{\beta_1})(\eta_\delta\circ u)\varphi\,dx.$$

Letting $\delta\to 0^+$ gives (3.5.12). The proof of (3.5.13) is similar. □

Lemma 3.5.3 (Moser's iteration)**.** *There exists $\Lambda>0$ such that*

$$\max\{\|u\|_{L^\infty(\Omega_1)},\|v\|_{L^\infty(\Omega_2)}\}\le\Lambda, \tag{3.5.15}$$

where

$$\Omega_1:=\mathbb{R}^N(u>1)\quad\textit{and}\quad\Omega_2:=\mathbb{R}^N(v>1),$$

for every $(u,v)\in\mathcal{D}^{1,p_1}(\mathbb{R}^N)\times\mathcal{D}^{1,p_2}(\mathbb{R}^N)$ solving problem (3.5.1)*.*

Proof. Given $w\in L^p(\Omega_1)$, we shall write $\|w\|_p$ in place of $\|w\|_{L^p(\Omega_1)}$ when no confusion can arise. Observe that $m(\Omega_1)<+\infty$ and define, provided $M>1$,

$$u_M(x):=\min\{u(x),M\},\quad x\in\mathbb{R}^N.$$

Choosing $\varphi:=u_M^{\kappa p_1+1}$, with $\kappa\ge 0$, in (3.5.12) gives

$$\begin{aligned}(\kappa p_1+1)&\int_{\Omega_1(u\le M)}u_M^{\kappa p_1}|\nabla u|^{p_1-2}\nabla u\nabla u_M\,dx\\ &\le M_1\int_{\Omega_1}a_1(1+v^{\beta_1})u_M^{\kappa p_1+1}\,dx.\end{aligned} \tag{3.5.16}$$

Through the Sobolev embedding theorem we have

$$\begin{aligned}&(\kappa p_1+1)\int_{\Omega_1(u\le M)}u_M^{\kappa p_1}|\nabla u|^{p_1-2}\nabla u\nabla u_M\,dx\\ &=(\kappa p_1+1)\int_{\Omega_1(u\le M)}(|\nabla u|u^\kappa)^{p_1}dx=\frac{\kappa p_1+1}{(\kappa+1)^{p_1}}\int_{\Omega_1(u\le M)}|\nabla u^{\kappa+1}|^{p_1}dx\\ &=\frac{\kappa p_1+1}{(\kappa+1)^{p_1}}\int_{\Omega_1}|\nabla u_M^{\kappa+1}|^{p_1}dx\ge C_1\frac{\kappa p_1+1}{(\kappa+1)^{p_1}}\|u_M^{\kappa+1}\|_{p_1^*}^{p_1}\end{aligned}$$

for appropriate $C_1 > 0$. By Remark 3.5.1, Hölder's inequality entails

$$\int_{\Omega_1} a_1(1+v^{\beta_1})u_M^{\kappa p_1+1}dx \leq \int_{\Omega_1} a_1(1+v^{\beta_1})u^{\kappa p_1+1}dx$$
$$\leq \left(\|a_1\|_{\xi_1} + \|a_1\|_{\zeta_1}\|v\|_{p_2^*}^{\beta_1}\right)\|u\|_{(\kappa p_1+1)\xi_1'}^{\kappa p_1+1}.$$

Hence, (3.5.16) becomes

$$\frac{\kappa p_1+1}{(\kappa+1)^{p_1}}\|u_M^{\kappa+1}\|_{p_1^*}^{p_1} \leq C_2\left(\|a_1\|_{\xi_1} + \|a_1\|_{\zeta_1}\|v\|_{p_2^*}^{\beta_1}\right)\|u\|_{(\kappa p_1+1)\xi_1'}^{\kappa p_1+1}.$$

Since $u(x) = \lim_{M\to\infty} u_M(x)$ a.e. in $\mathbb{R}^N$, using the Fatou lemma gives

$$\frac{\kappa p_1+1}{(\kappa+1)^{p_1}}\|u\|_{(\kappa+1)p_1^*}^{(\kappa+1)p_1} \leq C_2\left(\|a_1\|_{\xi_1} + \|a_1\|_{\zeta_1}\|v\|_{p_2^*}^{\beta_1}\right)\|u\|_{(\kappa p_1+1)\xi_1'}^{\kappa p_1+1},$$

namely

$$\|u\|_{(\kappa+1)p_1^*} \leq C_3^{\eta(\kappa)}\sigma(\kappa)\left(1+\|v\|_{p_2^*}^{\beta_1}\right)^{\eta(\kappa)}\|u\|_{(\kappa p_1+1)\xi_1'}^{\frac{\kappa p_1+1}{(\kappa+1)p_1}}, \tag{3.5.17}$$

where $C_3 > 0$, while

$$\eta(\kappa) := \frac{1}{(\kappa+1)p_1}, \quad \sigma(\kappa) := \left[\frac{\kappa+1}{(\kappa p_1+1)^{1/p_1}}\right]^{\frac{1}{\kappa+1}}.$$

Let us next verify that

$$(\kappa+1)p_1^* > (\kappa p_1+1)\xi_1' \quad \forall\, \kappa \in \mathbb{R}_0^+,$$

which clearly is equivalent to

$$\frac{1}{\xi_1} < 1 - \frac{\kappa p_1+1}{(\kappa+1)p_1^*}, \quad \kappa \in \mathbb{R}_0^+. \tag{3.5.18}$$

Indeed, the function $\kappa \mapsto \frac{\kappa p_1+1}{(\kappa+1)p_1^*}$ is increasing on $\mathbb{R}_0^+$ and tends to $\frac{p_1}{p_1^*}$, as $k \to \infty$. So, (3.5.18) holds true, because $\frac{1}{\xi_1} < 1 - \frac{p_1}{p_1^*}$; see Remark 3.5.1. Now, Moser's iteration can start. If there exists a sequence $(\kappa_n) \subseteq \mathbb{R}_0^+$ fulfilling

$$\lim_{n\to\infty} \kappa_n = +\infty, \quad \|u\|_{(\kappa_n+1)p_1^*} \leq 1 \;\; \forall\, n \in \mathbb{N}$$

then $\|u\|_{L^\infty(\Omega_1)} \leq 1$. Otherwise, with appropriate $\kappa_0 > 0$, we have

$$\|u\|_{(\kappa+1)p_1^*} > 1 \;\text{ for any }\; \kappa > \kappa_0, \;\text{ besides }\; \|u\|_{(\kappa_0+1)p_1^*} \leq 1. \tag{3.5.19}$$

Pick $\kappa_1 > \kappa_0$ such that $(\kappa_1 p_1+1)\xi_1' = (\kappa_0+1)p_1^*$, set $\kappa := \kappa_1$ in (3.5.17), and use (3.5.19) to arrive at

$$\begin{aligned} &\|u\|_{(\kappa_1+1)p_1^*} \\ &\leq C_3^{\eta(\kappa_1)}\sigma(\kappa_1)\left(1+\|v\|_{p_2^*}^{\beta_1}\right)^{\eta(\kappa_1)}\|u\|_{(\kappa_0+1)p_1^*}^{\frac{\kappa_1 p_1+1}{(\kappa_1+1)p_1}} \\ &\leq C_3^{\eta(\kappa_1)}\sigma(\kappa_1)\left(1+\|v\|_{p_2^*}^{\beta_1}\right)^{\eta(\kappa_1)}. \end{aligned} \tag{3.5.20}$$

Choose next $\kappa_2 > \kappa_0$ satisfying $(\kappa_2 p_1+1)\xi_1' = (\kappa_1+1)p_1^*$. From (3.5.17), written for $\kappa := \kappa_2$, as well as (3.5.19)–(3.5.20) it follows that

$$\begin{aligned} &\|u\|_{(\kappa_2+1)p_1^*} \\ &\leq C_3^{\eta(\kappa_2)}\sigma(\kappa_2)\left(1+\|v\|_{p_2^*}^{\beta_1}\right))^{\eta(\kappa_2)}\|u\|_{(\kappa_1+1)p_1^*}^{\frac{\kappa_2 p_1+1}{(\kappa_2+1)p_1}} \\ &\leq C_3^{\eta(\kappa_2)}\sigma(\kappa_2)\left(1+\|v\|_{p_2^*}^{\beta_1}\right)^{\eta(\kappa_2)}\|u\|_{(\kappa_1+1)p_1^*} \\ &\leq C_3^{\eta(\kappa_2)+\eta(\kappa_1)}\sigma(\kappa_2)\sigma(\kappa_1)\left(1+\|v\|_{p_2^*}^{\beta_1}\right)^{\eta(\kappa_2)+\eta(\kappa_1)}. \end{aligned}$$

By induction, we construct a sequence $(\kappa_n) \subseteq (\kappa_0, +\infty)$ enjoying the properties below:

$$(\kappa_n p_1 + 1)\xi_1' = (\kappa_{n-1} + 1)p_1^*, \quad n \in \mathbb{N}; \tag{3.5.21}$$

$$\|u\|_{(k_n+1)p_1^*} \le C_3^{\sum_{i=1}^{n} \eta(\kappa_i)} \prod_{i=1}^{n} \sigma(\kappa_i) \left(1 + \|v\|_{p_2^*}^{\beta_1}\right)^{\sum_{i=1}^{n} \eta(\kappa_i)} \tag{3.5.22}$$

for all $n \in \mathbb{N}$. A simple computation based on (3.5.21) yields

$$(\kappa_n + 1) \simeq (\kappa_0 + 1) \left(\frac{p_1^*}{p_1 \xi_1'}\right)^n \quad \text{as } n \to \infty, \tag{3.5.23}$$

where $\frac{p_1^*}{p_1\xi_1'} > 1$ due to (j_4) of Remark 3.5.1. Further, if $C_4 > 0$ satisfies

$$1 < \left[\frac{t+1}{(tp_1+1)^{1/p_1}}\right]^{\frac{1}{\sqrt{t+1}}} \le C_4, \quad t \in \mathbb{R}_0^+,$$

(cf. [44, p. 116]), then

$$\begin{aligned}\prod_{i=1}^{n} \sigma(\kappa_i) &= \prod_{i=1}^{n} \left(\frac{\kappa_i + 1}{(\kappa_i p_1 + 1)^{1/p_1}}\right)^{\frac{1}{\kappa_i+1}} \\ &= \prod_{i=1}^{n} \left[\left(\frac{\kappa_i + 1}{(\kappa_i p_1 + 1)^{1/p_1}}\right)^{\frac{1}{\sqrt{\kappa_i+1}}}\right]^{\frac{1}{\sqrt{\kappa_i+1}}} \le C_4^{\sum_{i=1}^{n} \frac{1}{\sqrt{\kappa_i+1}}}.\end{aligned}$$

Consequently, (3.5.22) becomes

$$\|u\|_{(k_n+1)p_1^*} \le C_3^{\sum_{i=1}^{n} \eta(\kappa_i)} C_4^{\sum_{i=1}^{n} \frac{1}{\sqrt{\kappa_i+1}}} \left(1 + \|v\|_{p_2^*}^{\beta_1}\right)^{\sum_{i=1}^{n} \eta(\kappa_i)}.$$

Observe that, by (3.5.23), we have that $\kappa_n + 1 \to +\infty$ and $\frac{1}{\kappa_n+1} \simeq \frac{1}{\kappa_0+1}\left(\frac{p_1\xi_1'}{p_1^*}\right)^n$. Moreover, (3.5.7) entails $\|v\|_{p_2^*} \le \rho$. Therefore, there exists a constant $C_5 > 0$ such that

$$\|u\|_{(\kappa_n+1)p_1^*} \le C_5 \quad \forall\, n \in \mathbb{N},$$

whence $\|u\|_{L^\infty(\Omega_1)} \le C_5$. Thus, in either case, $\|u\|_{L^\infty(\Omega_1)} \le \Lambda$, with $\Lambda := \max\{1, C_5\}$. A similar argument applies to v. □

Using (3.5.15) and taking into account the definition of sets Ω_i we immediately infer the following

Theorem 3.5.1. *Under assumptions* $(\mathrm{H}_{f,g})$ *and* (H_a)*, it follows that*

$$\max\{\|u\|_\infty, \|v\|_\infty\} \le \Lambda \tag{3.5.24}$$

for every weak solution $(u, v) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$ *to problem* (3.5.1)*. Here,* Λ *is given by Lemma 3.5.3.*

3.5.3 The regularized system

From Remark 3.5.1 we already know that $a_i \in L^{(p_i^*)'}(\mathbb{R}^N)$. Therefore, thanks to Minty-Browder's theorem (see [22, Theorem V.16]), the equation

$$-\Delta_{p_i} w_i = a_i(x) \quad \text{in} \quad \mathbb{R}^N \tag{3.5.25}$$

admits an unique solution $w_i \in \mathcal{D}^{1,p_i}(\mathbb{R}^N)$, $i = 1, 2$. Moreover, it is simple to prove that $w_i > 0$ and $w_i \in L^\infty(\mathbb{R}^N)$.

Indeed, testing (3.5.25) with $\varphi := w_i^-$ yields $w_i \geq 0$, because $a_i > 0$ by (H_a). Through the strong maximum principle we further obtain

$$\operatorname*{ess\,inf}_{B_r(x)} w_i > 0 \quad \text{for any } r > 0,\, x \in \mathbb{R}^N.$$

Hence, $w_i > 0$. Moser's iteration technique then produces $w_i \in L^\infty(\mathbb{R}^N)$.

Next, fix $\varepsilon \in \,]0,1[$ and define

$$(\underline{u}, \underline{v}) = \left([m_1(\Lambda+1)^{\alpha_1}]^{\frac{1}{p_1-1}} w_1, [m_2(\Lambda+1)^{\beta_2}]^{\frac{1}{p_2-1}} w_2 \right), \tag{3.5.26}$$

$$(\overline{u}_\varepsilon, \overline{v}_\varepsilon) = \left([M_1 \varepsilon^{\alpha_1}(1+\Lambda^{\beta_1})]^{\frac{1}{p_1-1}} w_1, [M_2 \varepsilon^{\beta_2}(1+\Lambda^{\alpha_2})]^{\frac{1}{p_2-1}} w_2 \right),$$

as well as

$$\mathcal{K}_\varepsilon := \left\{ (z_1, z_2) \in L^{p_1^*}(\mathbb{R}^N) \times L^{p_2^*}(\mathbb{R}^N) : \underline{u} \leq z_1 \leq \overline{u}_\varepsilon \,,\ \underline{v} \leq z_2 \leq \overline{v}_\varepsilon \right\}.$$

Obviously, $\mathcal{K}_\varepsilon$ is bounded, convex, closed in $L^{p_1^*}(\mathbb{R}^N) \times L^{p_2^*}(\mathbb{R}^N)$. Given $(z_1, z_2) \in \mathcal{K}_\varepsilon$, we write

$$\tilde{z}_i := \min\{z_i, \Lambda\}, \quad i = 1, 2. \tag{3.5.27}$$

On account of (3.5.27), hypothesis ($\mathrm{H}_{f,g}$) entails

$$\begin{aligned} a_1 m_1(\Lambda+1)^{\alpha_1} &\leq a_1 f(\tilde{z}_1 + \varepsilon, \tilde{z}_2) \leq a_1 M_1 \varepsilon^{\alpha_1}(1+\Lambda^{\beta_1}), \\ a_2 m_2(\Lambda+1)^{\beta_2} &\leq a_2 g(\tilde{z}_1, \tilde{z}_2 + \varepsilon) \leq a_2 M_2 (1+\Lambda^{\alpha_2}) \varepsilon^{\beta_2}. \end{aligned} \tag{3.5.28}$$

Moreover, since $a_i \in L^{(p_i^*)'}(\mathbb{R}^N)$, then the functions

$$x \mapsto a_1(x) f(\tilde{z}_1(x) + \varepsilon, \tilde{z}_2(x)), \quad x \mapsto a_2(x) g(\tilde{z}_1(x), \tilde{z}_2(x) + \varepsilon)$$

belong to $\mathcal{D}^{-1,p_1'}(\mathbb{R}^N)$ and $\mathcal{D}^{-1,p_2'}(\mathbb{R}^N)$, respectively. Consequently, by Minty-Browder's theorem again, there exists an unique weak solution $(u_\varepsilon, v_\varepsilon)$ of the problem

$$\begin{cases} -\Delta_{p_1} u = a_1(x) f(\tilde{z}_1(x) + \varepsilon, \tilde{z}_2(x)) & \text{in } \mathbb{R}^N, \\ -\Delta_{p_2} v = a_2(x) g(\tilde{z}_1(x), \tilde{z}_2(x) + \varepsilon) & \text{in } \mathbb{R}^N, \\ \ u_\varepsilon, v_\varepsilon > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{3.5.29}$$

Let $\mathcal{T} \colon \mathcal{K}_\varepsilon \to L^{p_1^*}(\mathbb{R}^N) \times L^{p_2^*}(\mathbb{R}^N)$ be defined by $\mathcal{T}(z_1, z_2) = (u_\varepsilon, v_\varepsilon)$ for every $(z_1, z_2) \in \mathcal{K}_\varepsilon$.

Lemma 3.5.4. *It holds $\underline{u} \leq u_\varepsilon \leq \overline{u}_\varepsilon$ and $\underline{v} \leq v_\varepsilon \leq \overline{v}_\varepsilon$. So, in particular, $\mathcal{T}(\mathcal{K}_\varepsilon) \subseteq \mathcal{K}_\varepsilon$.*

Proof. Via (3.5.26), (3.5.25), (3.5.29), and (3.5.28) we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}(-\Delta_{p_1}\underline{u}-(-\Delta_{p_1}u_\varepsilon))(\underline{u}-u_\varepsilon)^+dx\\
&=\int_{\mathbb{R}^N}\left(-\Delta_{p_1}(\{m_1(\Lambda+1)^{\alpha_1}\}^{\frac{1}{p_1-1}}w_1)-(-\Delta_{p_1}u_\varepsilon)\right)(\underline{u}-u_\varepsilon)^+dx\\
&=\int_{\mathbb{R}^N}a_1\left((m_1(\Lambda+1)^{\alpha_1}-f(\tilde z_1+\varepsilon,\tilde z_2)\right)(\underline{u}-u_\varepsilon)^+dx\le 0.
\end{aligned}$$

Furthermore, [108, Lemma A.0.5] gives

$$\begin{aligned}
&\int_{\mathbb{R}^N}(-\Delta_{p_1}\underline{u}-(-\Delta_{p_1}u_\varepsilon))(\underline{u}-u_\varepsilon)^+dx\\
&=\int_{\mathbb{R}^N}\left(|\nabla\underline{u}|^{p_1-2}\nabla\underline{u}-|\nabla u_\varepsilon|^{p_1-2}\nabla u_\varepsilon\right)\nabla(\underline{u}-u_\varepsilon)^+dx\ge 0.
\end{aligned}$$

This implies that

$$\int_{\mathbb{R}^N}\left(-\Delta_{p_1}\underline{u}-(-\Delta_{p_1}u_\varepsilon)\right)(\underline{u}-u_\varepsilon)^+dx=0,$$

which ensures $(\underline{u}-u_\varepsilon)^+=0$, i.e., $\underline{u}\le u_\varepsilon$. The remaining inequalities can be verified in a similar way. □

Lemma 3.5.5. *The operator $\mathcal{T}$ is continuous and compact.*

Proof. Pick a sequence $(z_{1,n},z_{2,n})\subseteq\mathcal{K}_\varepsilon$ such that

$$(z_{1,n},z_{2,n})\to(z_1,z_2)\quad\text{in}\quad L^{p_1^*}(\mathbb{R}^N)\times L^{p_2^*}(\mathbb{R}^N).$$

If $(u_n,v_n):=\mathcal{T}(z_{1,n},z_{2,n})$ and $(u,v):=\mathcal{T}(z_1,z_2)$, then

$$\int_{\mathbb{R}^N}|\nabla u_n|^{p_1-2}\nabla u_n\nabla\varphi\,dx=\int_{\mathbb{R}^N}a_1f(\tilde z_{1,n}+\varepsilon,\tilde z_{2,n})\varphi\,dx, \tag{3.5.30}$$

$$\int_{\mathbb{R}^N}|\nabla v_n|^{p_2-2}\nabla v_n\nabla\psi\,dx=\int_{\mathbb{R}^N}a_2g(\tilde z_{1,n},\tilde z_{2,n}+\varepsilon)\psi\,dx, \tag{3.5.31}$$

$$\int_{\mathbb{R}^N}|\nabla u|^{p_1-2}\nabla u\nabla\varphi\,dx=\int_{\mathbb{R}^N}a_1f(\tilde z_1+\varepsilon,\tilde z_2)\varphi\,dx,$$

$$\int_{\mathbb{R}^N}|\nabla v|^{p_2-2}\nabla v\nabla\psi\,dx=\int_{\mathbb{R}^N}a_2g(\tilde z_1,\tilde z_2+\varepsilon)\psi\,dx$$

for every $(\varphi,\psi)\in\mathcal{D}^{1,p_1}(\mathbb{R}^N)\times\mathcal{D}^{1,p_2}(\mathbb{R}^N)$. Set $\varphi:=u_n$ in (3.5.30). From (3.5.28) it follows, after using Hölder's inequality, that

$$\begin{aligned}
\|\nabla u_n\|_{p_1}^{p_1}&=\int_{\mathbb{R}^N}a_1f(\tilde z_{1,n}+\varepsilon,\tilde z_{2,n})u_n\,dx\\
&\le M_1\int_{\mathbb{R}^N}a_1\varepsilon^{\alpha_1}(1+\Lambda^{\beta_1})u_n\,dx\le C_\varepsilon\int_{\mathbb{R}^N}a_1u_n\,dx\\
&\le C_\varepsilon\|a_1\|_{(p_1^*)'}\|u_n\|_{p_1^*}\le c_{p_1}C_\varepsilon\|a_1\|_{(p_1^*)'}\|\nabla u_n\|_{p_1}\quad\forall\,n\in\mathbb{N},
\end{aligned}$$

where $C_\varepsilon:=M_1\varepsilon^{\alpha_1}(1+\Lambda^{\beta_1})$. This actually means that (u_n) is bounded in $\mathcal{D}^{1,p_1}(\mathbb{R}^N)$, because $p_1>1$. By (3.5.31), an analogous conclusion holds for (v_n). Along subsequences if necessary, we may thus assume that

$$(u_n,v_n)\rightharpoonup(u,v)\ \text{ in }\ \mathcal{D}^{1,p_1}(\mathbb{R}^N)\times\mathcal{D}^{1,p_2}(\mathbb{R}^N). \tag{3.5.32}$$

So, (u_n, v_n) converges strongly in $L^{q_1}(B_{r_1}) \times L^{q_2}(B_{r_2})$ for any $r_i > 0$ and any $1 \leq q_i \leq p_i^*$ whence, up to subsequences again,

$$(u_n, v_n) \to (u, v) \text{ a.e. in } \mathbb{R}^N. \tag{3.5.33}$$

Now, combining Lemma 3.5.4 with Lebesgue's dominated convergence theorem, we obtain

$$(u_n, v_n) \to (u, v) \text{ in } L^{p_1^*}(\mathbb{R}^N) \times L^{p_2^*}(\mathbb{R}^N), \tag{3.5.34}$$

as desired. Let us finally verify that $\mathcal{T}(\mathcal{K}_\varepsilon)$ is relatively compact. If $(u_n, v_n) := \mathcal{T}(z_{1,n}, z_{2,n})$, $n \in \mathbb{N}$, then (3.5.30)–(3.5.31) can be written. Hence, the previous argument yields a pair $(u, v) \in L^{p_1^*}(\mathbb{R}^N) \times L^{p_2^*}(\mathbb{R}^N)$ fulfilling (3.5.34), possibly along a subsequence. This completes the proof. □

Thanks to Lemmas 3.5.4–3.5.5, Schauder's fixed point theorem applies, and there exists $(u_\varepsilon, v_\varepsilon) \in \mathcal{K}_\varepsilon$ such that $(u_\varepsilon, v_\varepsilon) = \mathcal{T}(u_\varepsilon, v_\varepsilon)$. Through Theorem 3.5.1, we next arrive at

Theorem 3.5.2. *Under hypotheses* $(\mathrm{H}_{f,g})$ *and* (H_a), *for every* $\varepsilon > 0$ *small, problem* (3.5.3) *admits a solution* $(u_\varepsilon, v_\varepsilon) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$ *complying with* (3.5.24).

3.5.4 Existence of solutions

We are now ready to establish the main result.

Theorem 3.5.3. *Let* $(\mathrm{H}_{f,g})$ *and* (H_a) *be satisfied. Then,* (3.5.1) *has a weak solution* $(u, v) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$ *which is essentially bounded.*

Proof. Pick $\varepsilon := \frac{1}{n}$, with $n \in \mathbb{N}$ big enough. Theorem 3.5.2 gives a pair (u_n, v_n), where $u_n := u_{\frac{1}{n}}$ and $v_n := v_{\frac{1}{n}}$, such that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u_n|^{p_1-2} \nabla u_n \nabla \varphi \, dx &= \int_{\mathbb{R}^N} a_1 f\left(u_n + \frac{1}{n}, v_n\right) \varphi \, dx, \\
\int_{\mathbb{R}^N} |\nabla v_n|^{p_2-2} \nabla v_n \nabla \psi \, dx &= \int_{\mathbb{R}^N} a_2 g\left(u_n, v_n + \frac{1}{n}\right) \psi \, dx
\end{aligned} \tag{3.5.35}$$

for every $(\varphi, \psi) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$. Moreover, from Lemma 3.5.4 we have

$$0 < \underline{u} \leq u_n \leq \Lambda, \quad 0 < \underline{v} \leq v_n \leq \Lambda. \tag{3.5.36}$$

Thanks to $(\mathrm{H}_{f,g})$, (3.5.36), and (H_a), choosing $\varphi := u_n$, $\psi := v_n$ in (3.5.35) easily entails

$$\begin{aligned}
\|\nabla u_n\|_{p_1}^{p_1} &\leq M_1 \int_{\mathbb{R}^N} a_1 u_n^{\alpha_1+1}(1 + v_n^{\beta_1}) dx \leq M_1 \Lambda^{\alpha_1+1}(1 + \Lambda^{\beta_1}) \|a_1\|_1 \,, \\
\|\nabla v_n\|_{p_2}^{p_2} &\leq M_2 \int_{\mathbb{R}^N} a_2 (1 + u_n^{\alpha_2}) v_n^{\beta_2+1} dx \leq M_2 (1 + \Lambda^{\alpha_2}) \Lambda^{\beta_2+1} \|a_2\|_1,
\end{aligned}$$

whence both $(u_n) \subseteq \mathcal{D}^{1,p_1}(\mathbb{R}^N)$ and $(v_n) \subseteq \mathcal{D}^{1,p_2}(\mathbb{R}^N)$ are bounded. Along subsequences if necessary, we thus have (3.5.32)–(3.5.33). Let us next show that

$$(u_n, v_n) \to (u, v) \quad \text{strongly in} \quad \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N). \tag{3.5.37}$$

Testing the first equation in (3.5.35) with $\varphi := u_n - u$ yields

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p_1-2} \nabla u_n \nabla (u_n - u) dx = \int_{\mathbb{R}^N} a_1 f\left(u_n + \frac{1}{n}, v_n\right)(u_n - u) dx. \tag{3.5.38}$$

Observe that the right-hand side of (3.5.38) ges to zero as $n \to \infty$. Indeed, by ($\mathrm{H}_{f,g}$), (3.5.36) and (H_a) again,

$$\left| a_1 f\left(u_n + \frac{1}{n}, v_n\right)(u_n - u)\right| \le 2M_1 \Lambda^{\alpha_1+1}(1+\Lambda^{\beta_1})a_1 \quad \forall n \in \mathbb{N},$$

so that, recalling (3.5.33), Lebesgue's dominated convergence theorem applies. Through (3.5.38) we obtain $\lim_{n\to\infty} \langle -\Delta_{p_1} u_n, u_n - u\rangle = 0$. Likewise, $\langle -\Delta_{p_2} v_n, v_n - v\rangle \to 0$ as $n \to \infty$, and (3.5.37) directly follows from Proposition 3.5.2. On account of (3.5.35), and having in mind (3.5.37), the final step is to verify that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} a_1 f\left(u_n + \frac{1}{n}, v_n\right)\varphi\, dx = \int_{\mathbb{R}^N} a_1 f(u,v)\varphi\, dx, \tag{3.5.39}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} a_2 g\left(u_n, v_n + \frac{1}{n}\right)\psi\, dx = \int_{\mathbb{R}^N} a_2 g(u,v)\psi\, dx \tag{3.5.40}$$

for all $(\varphi,\psi) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$. If $\varphi \in \mathcal{D}^{1,p_1}(\mathbb{R}^N)$ then (j_1) in Remark 3.5.1 gives $a_1\varphi \in L^1(\mathbb{R}^N)$. Since, as before,

$$\left| a_1 f\left(u_n + \frac{1}{n}, v_n\right)\varphi\right| \le M_1 \Lambda^{\alpha_1+1}(1+\Lambda^{\beta_1})a_1|\varphi|, \quad n \in \mathbb{N},$$

assertion (3.5.39) stems from the Lebesgue dominated convergence theorem. The proof of (3.5.40) is similar at all.

It remains to prove that $u, v \to 0$ as $|x| \to +\infty$. We will prove the result only for u, since the arguments for v are completely similar. Consider again the first equation of (3.5.1) with $\varphi = u$. Since we already know that $v \in L^\infty(\mathbb{R}^N)$, it follows that

$$\int_{\mathbb{R}^N} |\nabla u|^{p_1-2}\nabla u \cdot \nabla\varphi dx \le C\int_{\mathbb{R}^N} a_1 u^{\alpha_1}\varphi dx, \quad \forall\, \varphi \in \mathcal{D}^{1,p}(\mathbb{R}^N), \tag{3.5.41}$$

where $C = C(\Lambda, M_1, \beta_1)$. For every $\kappa \ge 0$ and $n \in \mathbb{N}$ we choose the following test function $\varphi = u^{\kappa p_1+1}\eta_n^{p_1}$, where η_n is such that

$$\eta_n(x) = \begin{cases} 1 & \text{in } B_{n+1} := B_{1+\frac{1}{2^n}} \\ 0 & \text{in } B_n^c. \end{cases}$$

Clearly we have $\nabla\varphi = (\kappa p_1 + 1)\nabla u u^{\kappa p_1}\eta_n^{p_1} + p_1 u^{\kappa p_1+1}\eta_n^{p_1-1}\nabla\eta_n$, as well as $|\nabla\eta_n| \le C_1 2^n$, with $C_1 > 0$. Testing (3.5.41) with such φ we have

$$\begin{aligned}(\kappa p_1 + 1)\int_{\mathbb{R}^N} |\nabla u|^{p_1} u^{\kappa p_1}\eta_n^{p_1} dx + p_1\int_{\mathbb{R}^N} |\nabla u|^{p_1-2} u^{\kappa p_1+1}\eta_n^{p_1-1}\nabla\eta_n dx \\ \le C\int_{\mathbb{R}^N} a_1 u^{\alpha_1+1+\kappa p_1}\eta_n^{p_1} dx,\end{aligned}$$

which, passing to the absolute values, implies that

$$\begin{aligned}&(\kappa p_1 + 1)\int_{\mathbb{R}^N} |\nabla u|^{p_1} u^{\kappa p_1}\eta_n^{p_1} dx \\ &\le C\int_{\mathbb{R}^N} a_1 u^{\alpha_1+1+\kappa p_1}\eta_n^{p_1} dx + p_1\int_{\mathbb{R}^N} |\nabla u|^{p_1-1} u^{\kappa p_1+1}\eta_n^{p_1-1}|\nabla\eta_n| dx.\end{aligned} \tag{3.5.42}$$

Thanks to Young's inequality with exponents $p_1/(p_1-1)$ and p_1 it follows that

$$\begin{aligned}p_1\int_{\mathbb{R}^N} (|\nabla u|^{p_1-1}\eta_n^{p_1-1}u^{\kappa(p_1-1)})u^{\kappa+1}|\nabla\eta_n| dx \\ \le p_1\varepsilon\int_{\mathbb{R}^N} |\nabla u|^{p_1} u^{\kappa p_1}\eta_n^{p_1} dx + p_1 C_\varepsilon\int_{\mathbb{R}^N} u^{(\kappa+1)p_1}|\nabla\eta_n|^{p_1} dx,\end{aligned}$$

for every $\varepsilon > 0$. In particular, we choose $\varepsilon = \frac{\kappa p+1}{2p_1}$, and so equation (3.5.42) becomes

$$\frac{\kappa p_1 + 1}{2} \int_{\mathbb{R}^N} |\nabla u|^{p_1} u^{\kappa p_1} \eta_n^{p_1} dx \le C \int_{\mathbb{R}^N} a_1 u^{\alpha_1+1+\kappa p_1} \eta_n^{p_1} dx + p_1 C_\varepsilon \int_{\mathbb{R}^N} u^{(\kappa+1)p_1} |\nabla \eta_n|^{p_1} dx,$$

that is

$$\frac{\kappa p_1 + 1}{2(\kappa+1)^{p_1}} \int_{\mathbb{R}^N} |\nabla u^{\kappa+1}|^{p_1} \eta_n^{p_1} dx \le C \int_{\mathbb{R}^N} a_1 u^{\alpha_1+1+\kappa p_1} \eta_n^{p_1} dx + p_1 C_\varepsilon \int_{\mathbb{R}^N} u^{(\kappa+1)p_1} |\nabla \eta_n|^{p_1} dx.$$

Adding to both sides the positive quantity $\frac{\kappa p_1+1}{2(\kappa+1)^{p_1}} \int_{\mathbb{R}^N} u^{(\kappa+1)p_1} |\nabla \eta_n|^{p_1} dx$ and summarizing the constants implies

$$\frac{\kappa p_1 + 1}{(\kappa+1)^{p_1}} \int_{\mathbb{R}^N} |\nabla (u^{\kappa+1} \eta_n)|^{p_1} dx \le C \int_{\mathbb{R}^N} a_1 u^{\alpha_1+1+\kappa p_1} \eta_n^{p_1} dx + C_2 \int_{\mathbb{R}^N} u^{(\kappa+1)p_1} |\nabla \eta_n|^{p_1} dx.$$

Thanks to Sobolev's inequality we have

$$\begin{aligned} &\frac{1}{c^{p_1}} \frac{\kappa p_1 + 1}{(\kappa+1)^{p_1}} \left(\int_{\mathbb{R}^N} (u^{\kappa+1} \eta_n)^{p_1^*} dx \right)^{p_1/p_1^*} \\ &\qquad \le C \int_{\mathbb{R}^N} a_1 u^{\alpha_1+1+\kappa p_1} \eta_n^{p_1} dx + C_2 \int_{\mathbb{R}^N} u^{(\kappa+1)p_1} |\nabla \eta_n|^{p_1} dx, \end{aligned}$$

which, bearing in mind the definition of η_n, is equivalent to

$$\begin{aligned} &\frac{1}{c^{p_1}} \frac{\kappa p_1 + 1}{(\kappa+1)^{p_1}} \left(\int_{B_{n+1}} u^{(\kappa+1)p_1^*} dx \right)^{p_1/p_1^*} \\ &\qquad \le C \int_{B_n} a_1 u^{\alpha_1+1+\kappa p_1} dx + 2^{np_1} C_3 \int_{B_n} u^{(\kappa+1)p_1} dx. \end{aligned} \tag{3.5.43}$$

Applying Hölder's inequality with exponents ξ_1 and ξ_1' to the first term on the right-hand side gives

$$\int_{B_n} a_1 u^{\alpha_1+1+\kappa p_1} dx \le \|a_1\|_{\xi_1} \left(\int_{B_n} u^{(\alpha_1+1+\kappa p_1)\xi_1'} dx \right)^{1/\xi_1'}.$$

Moreover, choose $q = \frac{\kappa p_1+1}{\kappa p_1+1+\alpha_1}$ and apply Hölder's inequality with q and q' to the last integral of the inequality above (note that $q > 1$). This gives

$$\left(\int_{B_n} u^{(\alpha_1+1+\kappa p_1)\xi_1'} dx \right)^{1/\xi_1'} \le \left(\int_{B_n} u^{(\kappa p_1+1)\xi_1'} dx \right)^{\frac{\kappa p_1+1+\alpha_1}{(\kappa p_1+1)\xi_1'}} |B_n|^{\frac{-\alpha_1}{(\kappa p_1+1)\xi_1'}},$$

being $-\alpha_1 > 0$. Furthermore, let $s = \frac{(\kappa p_1+1)\xi_1'}{\kappa p_1}$, apply Hölder's inequality to the last term of (3.5.43) with exponents s and s', with $s > 1$, and take into account that $u \in L^\infty(\mathbb{R}^N)$

$$\begin{aligned} \int_{B_n} u^{(\kappa+1)p_1} dx = \int_{B_n} u^{\kappa p_1} u^{p_1} dx &\le \left(\int_{B_n} u^{(\kappa p_1+1)\xi_1'} dx \right)^{\frac{\kappa p_1}{(\kappa p_1+1)\xi_1'}} \left(\int_{B_n} u^{p_1 s'} dx \right)^{1/s'} \\ &\le |B_n| \Lambda^{p_1} \|u\|_{(\kappa p_1+1)\xi_1'}^{\kappa p_1}. \end{aligned}$$

Taking into account these inequalities it follows that

$$\begin{aligned} &\frac{1}{c^{p_1}} \frac{\kappa p_1 + 1}{(\kappa+1)^{p_1}} \left(\int_{B_{n+1}} u^{(\kappa+1)p_1^*} dx \right)^{p_1/p_1^*} \\ &\qquad \le C |B_n|^{\frac{-\alpha_1}{(\kappa p_1+1)\xi_1'}} \|a_1\|_{\xi_1} \|u\|_{L^{(\kappa p_1+1)\xi_1'}(B_n)}^{\kappa p_1+1+\alpha_1} + 2^{np_1} C_3 \Lambda^{p_1} |B_n| \|u\|_{L^{(\kappa p_1+1)\xi_1'}(B_n)}^{\kappa p_1}, \end{aligned}$$

which, summarizing the constants and taking into account once again that $u \in L^\infty(\mathbb{R}^N)$ gives

$$\|u\|^{(\kappa+1)p_1}_{L^{(\kappa+1)p_1^*}(B_{n+1})} \leq C_4 \left(|B_n|^{\frac{-\alpha_1}{(\kappa p_1+1)\xi_1'}} \|a_1\|_{\xi_1} \Lambda^{\kappa p_1+\alpha_1} + 2^{np_1}\Lambda^{\kappa p_1-1}|B_n| \right) \|u\|_{L^{(\kappa p_1+1)\xi_1'}(B_n)}.$$

that is

$$\begin{aligned} &\|u\|_{L^{(\kappa+1)p_1^*}(B_{n+1})} \\ &\leq \left[C_4 \left(|B_n|^{\frac{-\alpha_1}{(\kappa p_1+1)\xi_1'}} \|a_1\|_{\xi_1} \Lambda^{\kappa p_1+\alpha_1} + 2^{np_1}\Lambda^{\kappa p_1-1}|B_n| \right) \right]^{\frac{1}{(\kappa+1)p_1}} \|u\|^{\frac{1}{(\kappa+1)p_1}}_{L^{(\kappa p_1+1)\xi_1'}(B_n)}. \end{aligned} \tag{3.5.44}$$

We inductively construct a sequence (κ_n) such that $(\kappa_n p_1+1)\xi_1' = (\kappa_{n-1}+1)p_1^*$, for every $n \in \mathbb{N}_0$. Since it can be proved that $\kappa_n \simeq \left(\frac{p_1^*}{p_1\xi_1'}\right)^{n+1}$, where $\frac{p_1^*}{p_1\xi_1'} > 1$, it follows that $\kappa_n \to +\infty$ as $n \to +\infty$. Inserting $\kappa = \kappa_n$ in (3.5.44) we obtain

$$\begin{aligned} \|u\|_{L^{(\kappa_n+1)p_1^*}(B_{n+1})} \leq \Bigg[C_4 \Bigg(& |B_n|^{\frac{-\alpha_1}{(\kappa_n p_1+1)\xi_1'}} \|a_1\|_{\xi_1} \Lambda^{\kappa_n p_1+\alpha_1} \\ & + 2^{np_1}\Lambda^{\kappa_n p_1-1}|B_n| \Bigg) \Bigg]^{\frac{1}{(\kappa_n+1)p_1}} \|u\|^{\frac{1}{(\kappa_n+1)p_1}}_{L^{(\kappa_n p_1+1)\xi_1'}(B_n)}. \end{aligned} \tag{3.5.45}$$

Since

$$\lim_{n\to+\infty} \left[C_4 \left(|B_n|^{\frac{-\alpha_1}{(\kappa_n p_1+1)\xi_1'}} \|a_1\|_{\xi_1} \Lambda^{\kappa_n p_1+\alpha_1} + 2^{np_1}\Lambda^{\kappa_n p_1-1}|B_n| \right) \right]^{\frac{1}{(\kappa_n+1)p_1}} \in \mathbb{R},$$

there exists a constant $C_5 > 0$ such that

$$\left[C_4 \left(|B_n|^{\frac{-\alpha_1}{(\kappa_n p_1+1)\xi_1'}} \|a_1\|_{\xi_1} \Lambda^{\kappa_n p_1+\alpha_1} + 2^{np_1}\Lambda^{\kappa_n p_1-1}|B_n| \right) \right]^{\frac{1}{(\kappa_n+1)p_1}} \leq C_5, \tag{3.5.46}$$

for every $n \in \mathbb{N}_0$. We want to get a better estimate of inequality (3.5.45). To this end, let $\kappa_0 > 0$ such that $(\kappa_0 p_1+1)\xi_1' = p_1^*$. Inserting $\kappa = \kappa_0$ in (3.5.45) and taking into account (3.5.46) we have

$$\|u\|_{L^{(\kappa_0+1)p_1^*}(B_1)} \leq C_5 \|u\|^{\frac{1}{(\kappa_0+1)p_1}}_{L^{p_1^*}(B_0)}.$$

Let now $\kappa_1 > \kappa_0$ such that $(\kappa_1 p_1+1)\xi_1' = (\kappa_0+1)p_1^*$. Inserting $\kappa = \kappa_1$ in (3.5.45) we have

$$\begin{aligned} \|u\|_{L^{(\kappa_1+1)p_1^*}(B_2)} &\leq C_5 \|u\|^{\frac{1}{(\kappa_1+1)p_1}}_{L^{(\kappa_0+1)p_1^*}(B_1)} \\ &\leq C_5^{1+\frac{1}{(\kappa_1+1)p_1}} \|u\|^{\frac{1}{(\kappa_1+1)p_1}\frac{1}{(\kappa_0+1)p_1}}_{L^{p_1^*}(B_0)}. \end{aligned}$$

Proceeding by induction, we find $\kappa_n > \kappa_0$ for which it holds, after setting $a_n := \frac{1}{(\kappa_n+1)p_1}$, that

$$\|u\|_{L^{(\kappa_n+1)p_1^*}(B_{n+1})} \leq C_5^{1+a_n+a_na_{n-1}+\cdots+a_n\ldots a_1} \|u\|^{\prod_{i=0}^{n} a_n}_{L^{p_1^*}(B_0)}. \tag{3.5.47}$$

A simple computation shows that, if $\gamma := \frac{p_1\xi_1'}{p_1^*}$, then $a_n \simeq \gamma^{n+1}$, which implies that

$$a_{n-i}a_{n-i-1}\dots a_n \simeq \gamma^{n-i+1+n-i+1+1+\dots+n+1} = \gamma^{2in-\frac{i(i+1)}{2}+i} \quad \forall\, i \in \{1,\dots,n\},$$

and so

$$a_n + a_n a_{n-1} + \dots + a_n \dots a_1 \simeq \sum_{i=0}^{n} \gamma^{2in-\frac{i(i+1)}{2}+i}.$$

Moreover, after a changing of variables, it follows that

$$\sum_{n=0}^{N} \gamma^{2Nn-\frac{n(n+1)}{2}+n} \leq \sum_{n=0}^{N} \gamma^{\frac{3}{2}n(N+1)} \leq \sum_{n=0}^{N} \gamma^{n},$$

for N sufficiently large. From (3.5.47) we finally derive that

$$\|u\|_{L^{(\kappa_n+1)p_1^*}(B_{n+1})} \leq C_5^{\sum\limits_{i=0}^{n}\gamma^i} \|u\|_{L^{p_1^*}(B_0)}^{\prod\limits_{i=0}^{n}\gamma^n}.$$

Taking the limit as $n \to +\infty$ we find a constant $\tilde{C} > 0$ such that

$$\|u\|_{L^\infty(B_1)} \leq \tilde{C}\|u\|_{L^{p_1^*}(B_0)},$$

being B_1 the ball of radius 1 and B_0 the ball of radius 2. In what follows, we will write $B_2 := B_0$, for the reader's convenience. Let now $x \in \mathbb{R}^N$ be an arbitrary point. Then we have

$$\begin{aligned}
|u(x)| \leq \sup_{B_1(x)} &= \|u\|_{L^\infty(B_1)} \leq \tilde{C}\|u\|_{L^{p_1^*}(B_2(x))} \\
&= \tilde{C}\left(\int_{B_2(x)} u^{p_1^*}dx\right)^{1/p_1^*} \leq \tilde{C}\left(\int_{\mathbb{R}^N\setminus B_{\frac{|x|}{2}}(0)} u^{p_1^*}dx\right)^{1/p_1^*},
\end{aligned}$$

being $B_2(x) \subseteq \mathbb{R}^N \setminus B_{\frac{|x|}{2}}(0)$ whenever $|x| \geq 4$. Taking the limit as $|x| \to +\infty$, we obtain that $u(x) \to 0$, and so the thesis follows. A similar argument gives $v(x) \to 0$ as $|x| \to +\infty$. The proof is thus complete. □

3.6 Further developments

1. It is clear that the global bound obtained for solutions of (3.5.1), see inequality (3.5.15), is not sharp. Therefore, one task could be to see if it could be improved, maybe with more refined techniques.

2. One could investigate on further regularity properties of solutions obtained in Theorem 3.5.3, taking advantage of the regularity theory of Lieberman [84].

3. It could be interesting to see if the same existence results still hold if one considers a more general elliptic operator $\mathcal{A}$ than the p-Laplacian.

Bibliography

[1] R.A. Adams, *Sobolev spaces*, Academic Press, New York-London (1975). 46

[2] N.N. Akhmediev, A.V. Buryak and M. Karlsson, *Radiationless optical solitons with oscillating tails*, Optics Comm., **110**, pp. 540-544 (1994). 6

[3] C.O. Alves and F.J.S.A. Corrêa, *On the existence of positive solution for a class of singular systems involving quasilinear operators*, Appl. Math. Comput., **185**, pp. 727-736 (2007). 68

[4] C.O. Alves, J.V. Goncales and L.A. Maia, *Singular nonlinear elliptic equations in* $\mathbb{R}^N$, Abstr. Appl. Anal. **03**, pp. 411-423 (1998). 70

[5] O.H. Amann, T. von Karman and G.B. Woodruff, *The failure of the Tacoma Narrows bridge*, Federal Works Agency (1941). 5

[6] C.J. Amick and J.F. Toland, *Homoclinic orbits in the dynamic phase-space analogy of an elastic strut*, Eur. J. Appl. Math., **3**, pp. 97-114 (1992). 6

[7] G. Arioli and A. Szulkin, *A semilinear Schrödinger equation in the presence of a magnetic field*, Research reports in mathematics, **10** (2002). 42

[8] V.I. ArnoId and M.B. Sevryuk, *Oscillations and bifurcations in reversible systems*, in R. Z. Sachdeev, ed., Nonlinear Phenomena in Plasma Physics and Hydrodynamics, Mir, Moscow (1986). 10

[9] T. Aubin, *Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl., **55**, pp. 269-296 (1976). 38, 40

[10] T. Aubin, *Problèmes isopérimetriques et espaces de Sobolev*, J. Differential Geom., **11**, pp. 573-598 (1976). 36

[11] T. Aubin, *The scalar curvature*, Differential Geometry and Relativity, edited by Cahen and Flato, Reider (1976). 40

[12] T. Aubin, *Nonlinear analysis on manifolds. Monge-Ampére equations*, Grundlehren, **252**, Springer, New York-Heidelberg-Berlin (1982). 40

[13] E. Berchio, A. Ferrero, F. Gazzola and P. Karageorgis, *Qualitative behavior of global solutions to some nonlinear fourth order differential equations*, J. Differential Equations, **251**-10, pp. 2696-2727 (2011). 18

[14] J.B. van den Berg, *Dynamics and equilibria of fourth order differential equations*, Ph.D. thesis, Leiden University. 15

[15] J.B. van den Berg, *Branches of heteroclinic, homoclinic and periodic solutions in a fourth-order bi-stable system, a numerical study*, Master's thesis, Leiden University. 15

[16] F. Bernis, *Elliptic and parabolic semilinear problems without conditions at infinity*, Arch. Ration. Mech. Anal., **106**-3, pp. 217-241 (1989). 23

[17] J. Blat and K.J. Brown, *Bifurcation of steady-state solutions in predator-prey and competition systems*, Proc. Roy. Soc. Edinburgh Sect. A, **97**, pp 21-34 (1984). 64

[18] F. Bleich, C.B. McCullough, R. Rosecrans and G.S. Vincent, *The Mathematical Theory of Suspension Bridges*, U. S. Dept. of Commerce, Bureau of Public Roads (1950). 5

[19] D. Bonheure and L. Sanchez, *Heteroclinic orbits for some classes of second and fourth order differential equations*, in: Handbook of differential equations: ordinary differential equations., **3**, pp. 103-202 (2006). 34

[20] D. Bonheure, L. Sanchez, M. Tarallo and S. Terracini, *Heteroclinic connections between nonconsecutive equilibria of a fourth order differential equation*, Calc. Var. Partial Differential Equations **17**-4, pp. 341-356 (2003). 34

[21] E. Bour, *Théorie de la déformation des surfaces*, J. École Impériale Polytechnique, **19**, pp. 1-48 (1862). 2

[22] H. Brézis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York (2011). 79

[23] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., **36**, pp. 436-477 (1983). 38

[24] H. Brezis and L. Nirenberg, *Remarks on finding critical points*, Comm. Pure Appl. Math., **44**, pp. 939-963 (1991). 13

[25] B. Buffoni, *Multiplicity of homoclinic orbits in fourth-order conservative systems*, Variational and local methods in the study of Hamiltonian systems (Trieste, 1994), World Sci. Publishing, River Edge, NJ, pp. 129-136 (1995). 12, 15

[26] B. Buffoni, *Infinitely many large amplitude homoclinic orbits for a class of autonomous Hamiltonian systems*, J. Differential Equations, **121**, pp. 109-120 (1995). 6

[27] B. Buffoni and E. Séré, *A global condition for quasi-random behavior in a class of conservative systems*, Comm. Pure Appl. Math., **49**-3 pp. 285-305 (1996). 13, 15

[28] S. Čanić and B. Lee Keyfitz, *An elliptic problem arising from the unsteady transonic small disturbance equation*, J. Differential Equations, **125**, pp. 548-574 (1996). 66

[29] J. Chabrowski and A. Szulkin, *On the Schrödinger equation involving a critical Sobolev exponent and magnetic field*, Research reports in mathematics, **10** (2003). 42

[30] S. Chandrasekhar, *Hydrodynamic and hydromagnetic stability*, Dover Publications, Inc., New York (1961). vii

[31] Y. Chen and P.J. McKenna, *Traveling waves in a nonlinearly suspended beam: Theoretical results and numerical observations*, J. Differential Equations, **136**, pp. 325-355 (1997). 7, 13

[32] Y.S. Choi and E.H. Kim, *On the existence of positive solutions of quasilinear elliptic boundary value problems*, J. Differential Equations, **155** pp. 423-442 (1999). 66

[33] Y.S. Choi, A.C. Lazer and P.J. McKenna, *On a singular quasilinear anisotropic elliptic boundary value problem*, Trans. Amer. Math. Soc., **347**, pp. 2633-2641 (1995). 66

[34] Y.S. Choi and P.J. McKenna, *On a singular quasilinear anisotropic elliptic boundary value problem, II*, Trans. Amer. Math. Soc., **350**, pp. 2925-2937 (1998). 66

[35] P. Clement, J. Fleckinger, E. Mitidieri and F. de Thelin, *Existence of positive solutions for a nonvariational quasilinear elliptic system*, J. Differential Equations, **166**, pp. 455-477 (2000). 66

[36] E.A. Coddington and N. Levinson, *Theory o f Ordinary Differential Equations*, McGraw-Hill, New York (1955). 10

[37] V. Coti Zelati and P.H. Rabinowitz, *Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials*, J. Amer. Math. Soc., **4**-4, pp. 693-727 (1991). 15

[38] C. Conley, *Isolated invariant sets and the Morse index*, CMBS Lecture Notes, **38**, Amer. Math. Soc. (1978). 10

[39] M.G. Crandall, P.H. Rabinowitz and L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations, **2**-2, pp. 193-222 (1997). 66

[40] R. Dalmasso, *Existence and uniqueness of positive solutions of semilinear elliptic systems*, Nonlinear Anal., **39**, pp. 559-568 (2000). 66

[41] G.T. Dee and W. van Saarloos, *Bistable systems with propagating fronts leading to pattern formation*, Phys. Rev. Lett., **60**, pp. 2641-2644 (1988). vii, 3

[42] R.L. Devaney, *Homoclinic orbits in Hamiltonian systems*, J. Differential Equations, **21**-2 pp. 431-438 (1976). 15

[43] R.L. Devaney, *Blue sky catastrophes in reversible and Hamiltonian systems*, Indiana Univ. Math. J., **26**, pp. 247-263 (1977). 10

[44] P. Drábek, A. Kufner and F. Nicolosi, *Quasilinear Elliptic Equations with Degenerations and Singularities*, Walter de Gruyter & Co., Berlin (1997). viii, 44, 45, 47, 78

[45] P.G. Drazin and R.S. Johnson, *Solitons: An Introduction*, Cambridge University Press, Cambridge, UK (1989). 3

[46] S. El Manouni, K. Perera and R. Shivaji, *On singular quasimonotone (p,q)-Laplacian systems*, Proc. Roy. Soc. Edinburgh Sect. A, **142**, pp. 585-594 (2012). 68

[47] I. Ekeland and J.M. Lasry, *On the number of periodic trajectories for Hamiltonian flows on a convex energy surface*, Ann. Math., **112**, pp. 283-319 (1980). 12

[48] H. Federer and W. Fleming, *Normal and integral currents*, Ann. Math., **72**, pp. 458-520 (1960). 36

[49] V. Ferreira, Jr. and E. Moreira dos Santos, *On the finite space blow up of the solutions of the Swift-Hohenberg equation*, Calc. Var. Partial Differential Equations, **54**-1, pp. 1161-1182 (2015). 18

[50] J. Frenkel and T. Kontorova, *On the theory of plastic deformation and twinning*, Izvestiya Akademii Nauk SSSR, Seriya Fizicheskaya, **1**, pp. 137-149 (1939). 2

[51] J. Garcia Azorero and I. Peral Alonso, *Some results about the existence of a second positive solution in a quasilinear critical problem*, Indiana Univ. Math. J., **43**- 3, pp. 941-957 (1994). 39, 45

[52] L. Gasiński and N.S. Papageorgiou, *Exercises in Analysis. Part 1: Nonlinear Analysis*, Springer, Heidelberg, (2014). 56

[53] F. Gazzola and P. Karageorgis, *Refined blow-up results for nonlinear fourth order differential equations*, Commun. Pure Appl. Anal., **14**-2, p. 677-693 (2015). 18

[54] M. Ghergu, *Steady-state solutions for Gierer-Meinhardt type systems with Dirichlet boundary condition.* Trans. Amer. Math. Soc., **361**-8, pp. 3953-3976 (2009). 67

[55] J. Giacomoni, J. Hernandez and A. Moussaoui, *Quasilinear and singular systems: the cooperative case*, Contemp. Math, 540, pp. 79-94 (2011). 68

[56] J. Giacomoni, J. Hernandez and P. Sauvy, *Quasilinear and singular elliptic systems*, Advances Nonl. Anal., **540**, pp. 79-94 (2012). 68

[57] A. Gierer and H. Meinhardt, *A theory of biological pattern formation*, Kybernetik, **12**, pp. 30-39 (1972). ix, 66, 67

[58] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag Berlin Heidelberg New York (1998). 42, 43

[59] G. Gompper and M. Schick, *Self-assembling amphiphilic systems*, Phase Transitions and Critical Phenomena, **16**, C. Domb and J.L. Lebowitz eds., Academic Press, London, pp. 1-176 (1994). 7

[60] M. Guedda and L. Veron, *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal., **13**-8, pp. 879-902 (1989). 38

[61] P. Habets, L. Sanchez, M. Tarallo and S. Terracini, *Heteroclinics for a class of fourth order conservative differential equations*, CD ROM Proceedings, Equadiff 10 Prague, pp. 203-216 (2001). 34

[62] D.D. Hai, *Existence and uniqueness of solutions for quasilinear elliptic systems*, Proc. Amer. Math. Soc., **133**-1, pp. 223-228 (2004). 66

[63] P. Hartman, *Ordinary differential equations*, John Wiley & Sons Inc., New York (1964). 9

[64] R.W. Hasse, *A general method for the solution of nonlinear soliton and kink Schrödinger equations*, Z. Phys. B, **37**, pp. 83-87 (1980). 42

[65] H. Hofer and J.F. Toland, *Homoclinic, heteroclinic and periodic solutions for indefinite Hamiltonian systems*, Math. Ann., **268**, pp. 387-403 (1984). 12

[66] G.W. Hunt, H.M. Bolt and J.M.T. Thompson, *Structural localization phenomena and the dynamical phase-space analogy*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **425**, pp. 245-267 (1989). 6

[67] G.W. Hunt and M.K. Wadee, *Comparative Lagrangian formulations for localized buckling*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **434**, pp. 485-502 (1991). 6

[68] W.D. Kalies, J. Kwapisz and R.C.A.M. van der Vorst, *Homotopy classes for stable connections between Hamiltonian saddle-focus equilibria*, Comm. Math. Phys., **193**, pp. 337-371 (1998). 13, 15

[69] W.D. Kalies and R.C.A.M. van der Vorst, *Multitransition homoclinic and heteroclinic solutions to the extended Fisher-Kolmogorov equation*, J. Differential Equations, **131**, pp. 209-228 (1996). 13, 15

[70] D. Kandilakis and N. Sidiropoulos, *Existence and uniqueness results of positive solutions for nonvariational quasilinear elliptic system*, Electron. J. Differential Equations, **84**, pp. 1-6 (2006). 66

[71] P.M. Kareiva, *Local movement in herbivorous insects: applying a passive diffusion model to mark-recapture field experiments*, Oecologia (Berlin), **57**, pp. 322-327 (1983). 63

[72] J.P. Keener, *Activators and inhibitors in pattern formation*, Stud. Appl. Math., **59**, pp. 1-23 (1978). 67

[73] K. Kirchgässner, *Nonlinear resonant surface waves and homoclinic bifurcation*, Adv. Appl. Math., **26**, pp. 135-181 (1988). 6

[74] S. Kurihura, *Large-amplitude quasi-solitons in superfluid films*, J. Phys. Soc. Japan, **50**, pp. 3262-3267 (1981). 42

[75] E.W. Laedke, K.H. Spatschek and L. Stenflo, *Evolution theorem for a class of perturbed envelope soliton solutions*, J. Math. Phys., **24**, pp. 2764-2769 (1983). 42

[76] H. Lange, B. Toomire and P.F. Zweifel, *Time-dependent dissipation in nonlinear Schrödinger systems*, J. Math. Phys., **36**, pp. 1274-1283 (1995). 42

[77] A.C. Lazer and P.J. McKenna, *Large-amplitude oscillations in suspension bridges: Some new connections with nonlinear analysis*, SIAM Rev., **32**, pp. 537-578 (1990). vii, 5, 17

[78] A.C. Lazer and P.J. McKenna, *On a singular nonlinear elliptic boundary value problem*, Proc. Amer. Math. Soc., **111**, pp. 721-730 (1991). 66

[79] A.C. Lazer and P.J. McKenna, *On travelling waves in a suspension bridge model as the wave speed goes to zero*, Nonlinear Anal., **74**-12, pp. 3998-4001 (2011). vii, 17, 19

[80] A. Lê, *Eigenvalue problems for the p-Laplacian*, Nonlinear Anal., **64**-5, pp. 1057-1099 (2006). 44

[81] J. Lega, J.V. Moloney and A.C. Newell, *Swift-Hohenberg equation for lasers*, Phys. Rev. Lett., **73**-22, pp. 2978-2981 (1994). vii, 4

[82] G. Leoni, *A first course in Sobolev Spaces*, American Mathematical Society (2009). 37

[83] E.H. Lieb and M. Loss, *Analysis. Second edition*, Grad. Stud. Math., **14**, Amer. Math. Soc., Providence, RI (2001). 71

[84] G.M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations, **16**, pp. 311-361 (1991). 55, 57, 58, 85

[85] P.L. Lions, *On the existence of positive solutions of semilinear elliptic equations*, SIAM Rev., **24**, pp. 441-467 (1982). 38

[86] P.L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case, Part 1*, Rev. Mat. Iberoamericana, **1**-1, pp. 145-201 (1985). 42

[87] J. López-Gómez and R. Pardo, *Coexistence regions in Lotka-Volterra models with diffusion*, Nonlinear Anal., **19**, pp. 11-28 (1992). 64

[88] A.J. Lotka, *Undamped oscillations derived from the law of mass action*, J. Amer. Chem. Soc., **42**, pp. 1595-1599 (1920). 61

[89] T.R. Malthus, *An essay on the Principal of Population*, Penguin Books (1970). Originally published in 1798. 60

[90] V.G. Makhankov and V.K. Fedyanin, *Nonlinear effects in quasi-one-dimensional models of condensed matter theory*, Phys. Rep., **104**-1, pp. 1-86 (1984). 42

[91] S.A. Marano, G. Marino and A. Moussaoui, *Singular quasilinear elliptic systems in* $\mathbb{R}^N$, submitted to Annali di Matematica Pura e Applicata, 13 pp. (2018). viii, ix, 70

[92] G. Marino and S. Mosconi, *Existence and asymptotic behavior of nontrivial solutions to the Swift-Hohenberg equation*, J. Differential Equations, **263**-12, pp. 8581-8605 (2017). vii, 19

[93] G. Marino and P. Winkert, *Moser iteration applied to elliptic equations with critical growth on the boundary*, Nonlinear Anal., **180**, pp. 154-169 (2019). viii, 44

[94] V. Maz'ya, *Classes of domains and imbedding theorems for function spaces*, (Russian). Dokl. Akad. Nauk, **133**, pp. 527-530 (1960); English transl.: Sov. Math., Dokl., **1**, pp. 882-885 (1960). 36

[95] P.J. McKenna and W. Walter, *Nonlinear oscillations in a suspension bridge*, Arch. Ration. Mech. Anal., **87**, pp. 167-177 (1987). vii, 5, 17

[96] P.J. McKenna and W. Walter, *Traveling waves in a suspension bridge*, SIAM J. Appl. Math., **50**, pp. 703-715 (1990). vii, 5, 17

[97] S. Mosconi and S. Santra, *On the existence and non-existence of bounded solutions for a fourth order ODE*, J. Differential Equations, **255**-11, pp. 4149-4168 (2013). 17, 19, 20, 23

[98] D. Motreanu and A. Moussaoui, *Existence and boundedness of solutions for a singular cooperative quasilinear elliptic system*, Complex Var. Elliptic Equ., pp. 285-296 (2013). 68

[99] D. Motreanu and A. Moussaoui, *A quasilinear singular elliptic system without cooperative structure*, Acta Math. Sci. Ser. B Engl. Ed., **34**, pp. 905-916 (2014). 68

[100] D. Motreanu and A. Moussaoui, *An existence result for a class of quasilinear singular competitive elliptic systems*, Appl. Math. Lett., **38**, pp. 33-37 (2014). 68

[101] A. Moussaoui, B. Khodja and S. Tas, *A singular Gierer-Meinhardt system of elliptic equations in* $\mathbb{R}^N$, Nonlinear Anal., **71**, pp. 708-716 (2009). ix, 70

[102] N.S. Papageorgiou and V.D. Rădulescu, *Nonlinear nonhomogeneous Robin problems with superlinear reaction term*, Adv. Nonlinear Stud., **16**-4, pp. 737-764 (2016). 45

[103] L.A. Peletier and W.C. Troy, *Spatial patterns described by the extended Fisher- Kolmogorov (EFK) equation: kinks*, Differential Integral Equations, **8**-6, pp.1279-1304 (1995). 9, 11

[104] L.A. Peletier and W.C. Troy, *A topological shooting method and the existence of kinks of the extended Fisher-Kolmogorov equation*, Topol. Methods Nonlinear Anal., **6**-2, pp. 331-355 (1995). 9, 11, 15

[105] L.A. Peletier and W.C. Troy, *Chaotic spatial patterns described by the extended Fisher-Kolmogorov equation*, J. Differential Equations, **129**-2, pp. 458-508 (1996). 9

[106] L.A. Peletier and W.C. Troy, *Spatial patterns*, Progr. Nonlinear Differential Equations Appl., **45** (2001). 7, 9, 10, 11, 14, 15, 16

[107] L.A. Peletier, W.C. Troy and R.C.A.M. van der Vorst, *Stationary solutions of a fourth order nonlinear diffusion equation*, Differ. Equ., **31**, pp. 301-314 (1995). 13, 15

[108] I. Peral, *Multiplicity of Solutions for the p-Laplacian*, ICTP Lecture Notes of the Second School of Nonlinear Functional Analysis and Applications to Differential Equations, Trieste (1997). 73, 80

[109] D. Pierotti and S. Terracini, *On a neumann problem with critical exponent and critical nonlinearity on the boundary*, Comm. Partial Differential Equations, **20**, pp. 1155-1187 (1995). 39

[110] S.I. Pohozaev, *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Soviet Math. Doklady, **6**, pp. 1408-1411 (1965). 38

[111] Y. Pomeau and P. Manneville, *Wavelength selection in cellular flows*, Phys. Lett. A, **75**, pp. 296-298 (1980). 4

[112] P. Pucci and J. Serrin, *The Maximum Principle*, Birkhäuser Verlag, Basel (2007). 55

[113] P.H. Rabinowitz, *Periodic solutions of Hamiltonian systems*, Comm. Pure Appl. Math., **31**, pp. 157-184 (1978). 12

[114] P.H. Rabinowitz, *Periodic solutions of a Hamiltonian system on a prescribed energy surface*, J. Differential Equations, **33**, pp. 336-352 (1979). 12

[115] P. Radu, D. Toundykov and J. Trageser, *Finite time blow-up in nonlinear suspension bridge models*, J. Differential Equations, **257**-11, pp. 4030-4063 (2014). 18

[116] J.W.S. Rayleigh, *On the convective currents in a horizontal layer of fluid when the higher temperature is on the under side*, Phil. Mag., **32**, pp. 529-546 (1916). 4

[117] V. Rottschllfer and A. Doelman, *On the transition from the Ginzburg- Landau equation to the extended Fisher-Kolmogorov equation*, Phys. D, **118**, pp. 261-292 (1998). vii, 4

[118] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom., **20**, pp. 479-495 (1984). 40

[119] A.C. Scott, *Active and Nonlinear Wave Propagation in Electronics*, Wiley-Interscience, New York (1970). 3

[120] E. Séré, *Existence of infinitely many homoclinic orbits in Hamiltonian systems*, Math. Z., **209**-1, pp. 27-42 (1992). 15

[121] N. Shigesada, K. Kawasaki and E. Teramoto, *Spatial segregation of interacting species*, J. Theoret. Biology, **79**, pp. 83-99 (1979). 63

[122] N. Shigesada and J. Roughgarden, *The role of rapid dispersal in the population dynamics of competition*, Theor. Popul. Biol., **21**, pp. 353-372 (1982). 63

[123] D. Smets and J.B. van der Berg, *Homoclinic Solutions for Swift-Hohenberg and Suspension Bridge Type Equations*, J. Differential Equations, **184**-1, pp. 78-96 (2002). 27

[124] S.L. Sobolev, *On a theorem of functional analysis* (in Russian), Mat. Sb., **4** (1938). 36

[125] S.L. Sobolev, *Applications of functional analysis in mathematical physics*, Amer. Math. Soc. (1963). 36

[126] J.J. Stoker, *Water Waves*, Interscience Publ., New York (1957). 6

[127] C.A. Stuart, *Existence theorems for a class of nonlinear integral equations*, Math. Z., **137**, pp. 49-66 (1974). 66

[128] M. Struwe, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z., **187**, pp. 511-517 (1984). 39

[129] M. Struwe, *Variational Methods*, Springer-Verlag, Berlin (2008). viii, 40, 45

[130] J.B. Swift and P.C. Hohenberg, *Hydrodynamic fluctuations at the convective instability*, Phys. Rev. A, **15**, pp. 319-328 (1977). vii, 4

[131] G. Talenti, *Best constants in Sobolev inequality*, Ann. Mat. Pura Appl., **110**, pp. 353-372 (1976). 36

[132] S.D. Taliaferro, *A nonlinear singular boundary value problem*, Nonlinear Anal., **3**, pp. 897-904 (1979). 66

[133] M. Tlidi, M. Georgiou and P. Mandel, *Transverse patterns in nascent optical bistability*, Phys. Rev. A, **48**-2, pp. 4506-4609 (1993). vii, 4

[134] P. Tolksdorf, *On the Dirichlet problem for quasilinear equations in domain with conical boundary points*, Comm. Partial Differential Equations., **8**, pp. 773-817 (1983). 38

[135] H. Triebel, *Theory of function spaces*, Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig (1983). 46

[136] H. Triebel, *Theory of function spaces II*, Basel, Birkhäuser Verlag (1992). 46

[137] N. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci., **3**, pp. 265-274 (1968). viii, 38, 40

[138] A. Turing, *The chemical basis of morphogenesis*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **237**, pp. 37-72 (1952). 63

[139] A. Vanderbauwhede, *Subharmonic branching in reversible systems*, SIAM J. Math. Anal., **21**, pp. 954-979 (1990). 10

[140] A. Vanderbauwhede and B. Fiedler, *Homoclinic period blowup in reversible and conservative systems*, Z. Angew. Math. Phys., **43**, pp. 292-318 (1992). 10

[141] P.-F. Verhulst, *Notice sur la loi que la population suit dans son accroissement*, Corr. Math. et Phys., **10**, pp. 113-121(1838). 60

[142] P.-F. Verhulst, *Recherche mathémathiques sur le loi d'accroissement de la population*, Nouveau Mémoires de l'Académie Royale des Sciences et Belles Lettres de Bruxelles, **18**, pp. 3-38 (1845). 60

[143] V. Volterra, *Variazioni e fluttuazioni del numero d'individui in specie animali conviventi*, Mem. Acad. Lincei., **2**, pp. 31-113 (1926). 61

[144] X.-J. Wang, *Neumann problems of semilinear elliptic equations involving critical Sobolev exponents*, J. Differential Equations, **93**, pp. 283-310 (1991). 39

[145] T. Ważewski, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires*, Ann. Soc. Polon. Math., **20**, pp. 279-313 (1947). 10

[146] A. Weinstein, *Periodic orbits for convex Hamiltonian systems*, Ann. Math., **108**, pp. 507-518 (1978). 12

[147] A. Weinstein, *On the hypotheses of Rabinowitz Periodic orbit theorem*, J. Differential Equations, **33**, pp. 353-358 (1979). 12

[148] G.B. Whitham, *Linear and nonlinear waves.*, J. Wiley, New York (1974). 6

[149] P. Winkert, L^∞*-estimates for nonlinear elliptic Neumann boundary value problems*, NoDEA Nonlinear Differential Equations Appl., **17**-3, pp. 289-302 (2010). 48

[150] P. Winkert, *On the boundedness of solutions to elliptic variational inequalities*, Set-Valued Var. Anal., **22**-4, pp. 763-781 (2014). 46, 48

[151] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka J. Math. **12**, pp. 21-37 (1960). 40

[152] E. Zeidler, *Beiträge zur Theorie und Praxis freier Randwertaufgaben.*, Akademie-Verlag, Berlin (1971). 6

[153] E. Zeidler, *Bifurcation theory and permanent waves*, in Applications of Bifurcation Theory (P. Rabinowitz, ed.), Academic Press, pp. 203-223 (1977). 6